Exercises in Physics of Climate
A workbook companion to
Principles of Planetary Climate

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Preface

This document contains exercises to be used in conjunction with the text *Principles of planetary climate.*. A subset of the problems have been included in the print version of the book, in the Workbook sections for each chapter. This file is being maintained separately to provide room for additional problems that did not fit in the print version. The instructor can also make use of the LaTeX source file to copy and paste problems into individual problem sets. The LaTeX problem label tags provide unique problem identifiers, which make it easy to identify the corresponding solution write-ups.

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Chapter 1

Big Questions

1.1 Workbook

1.1.1 Your Computational Toolkit

The following problems introduce the computer and numerical analysis skills needed to do the problems in the Workbook sections of each chapter. There are a few specialized techniques which will be introduced in later chapters, but these basic skills will see the student a very long way indeed, if thoroughly mastered.

The numerical algorithms are introduced below by name, but no attempt has been made to describe or motivate the algorithm here. For that, the student should consult the book *Numerical Recipes*. Should you write your own routines, or use ready-made routines from a library of some sort? Ideally, one should do a bit of both – writing a basic implementation to see what is going on, and then eventually going over to a full-featured professionally written implementation for heavy-duty work. If time is short, it is reasonable to cut corners by going directly to the packaged routines, so long as one has some idea about what is going on inside the black box. In any event, the intent is that after completion of the problems in this section, the reader will be able to carry out all the necessary numerical operations.

Data referred to here and elsewhere in the Workbook sections of this book is provided as part of the online supplement. it is presumed that the instructor has downloaded the data and put it in a place that is available to everybody who needs it. The reader who is doing the problems on his or her own can simply download the data and put it wherever convenient.

Python tips: First, of course, you should learn some basic Python programming. This can be done through the tutorial material provided as part of the online supplement, or through information found at the web site `python.org`. The Python implementations are for the most part done in an object-oriented fashion. This may sound fancy and high-tech, but it's really quite simple. A kind of object is defined by a `class`, which allows one to create objects that perform some kind of task. This could be integration of a differential equation, evaluation of a definite integral or just about anything. If one provides the needed specifications, an object can be called like a function, indexed like an array, or even have arithmetic operations defined on it; an object can store data as well as means to manipulate the data. Objects make it easy to substitute alternate algorithms that perform the same function, without having to do major surgery on the rest of the program. Object-oriented programming is a very versatile technique, which is implemented in all modern programming languages, including even `MATLAB` and `FORTRAN 90`. Python provides far and away the gentlest introduction to object oriented programming, however. This style of programming has not yet pervaded scientific simulation, but the reader would do well to gain some facility with it, regardless of which programming language is the language of choice.

For the Python user, the basic courseware tools are implemented in the module `ClimateUtilities.py`. This module contains utilities for data input and output, graphics, and numerical analysis. Since the objects and functions defined in this module are so extensively used in solving the Workbook problems, you should normally import it into the Python interpreter using

```
from ClimateUtilities import *
```

Pay particularly close attention to the `Curve` class defined in `ClimateUtilities`. It allows you to create `Curve` objects that make it easy to do input and output of tabular data without dealing with low-level details, and also facilitates graphics and data manipulation. As with all Python objects, you can find out what features the `Curve` class has by typing `dir(Curve)`, and you can access built-in documentation by typing `help(Curve)`.

The other main courseware modules are `phys.py`, and `planets.py`. `phys.py` provides universal physical constants, thermodynamic properties and frequently-used functions for computing various physical quantities. `planets.py` contains data on selected astronomical bodies. The use of these two modules will be introduced in the workbook sections of later chapters.

The online supplement provides Python example scripts showing how to use the tools in `ClimateUtilities` to solve problems of the type introduced below. It also provides a number of numerical analysis tutorials which introduce the basic ideas behind the key algorithms. These can be used to supplement the deeper discussion provided in *Numerical Recipes*.

Problem 1.1.1 *Plotting a function*

Make a graph of Eq. **??** showing the evolution of Solar luminosity over time. As the vertical axis, you may use the luminosity relative to its present value. Show values out to twice the current age of the Sun. On the same plot show a straight-line approximation to the function, which yields the same values as Eq. **??** at $t = 0$ and $t = t_\odot$.

There are two different ways to go about making the plot. One alternative is to generate the data to be plotted in the same software package you use for plotting, and then carry out the operations (commands, menu choices, etc.) needed to carry out the plot. The second alternative is to generate the data in one programming environment, write it out to a text file, and then read the file into the plotting function of your choice. This can be handy if you want to use commercial plotting software that produces professional-looking figures.

Python Tips: Using the courseware in Python, you do this by making a list of arguments to the function, making a list of corresponding function values, installing each list in a `Curve` object, and then plotting. If the curve object is `MyCurve`, you plot using `plot(MyCurve)`. Check the examples in the supplementary material to see how to set plot options such as axis labels, titles, etc. As an alternative to plotting within Python, you can use the `dump` method of the `Curve` object to write out the data to a text file, which you can then read into the plotting software of your choice. For example, `MyCurve.dump('out.txt')` writes the data in `MyCurve` to a text file called `out.txt`.

Problem 1.1.2 *Reading and writing tabular data*

Tabular data is a very common form of scientific dataset. A tabular dataset consists of a number of columns of identical length. Each column represents a quantity of interest, and each row represents a set of the quantities pertaining to a given "sample." The "sample" could represent some individual chunk of rock, but more commonly, it will represent some point in space or time. For example, we could have a three column data set in which the first column is time, the second is air temperature and the third is the moisture content of the air. Or, the first column could be height above the surface of a planet while the second column is air pressure and the third is air density.

When the amount of data involved is not huge, it is convenient to store tabular data in the form of plain human-readable text files. Since these files can be examined in any text editor, it is easy to determine their structure and how to read the data into just about any software. There are still a few variations on the theme that one must keep in mind. First, there must always be some scheme to separate one column from another. This is usually done using a delimiter character, which is often a tab character, but could also be an arbitrary number of blank spaces or a comma. Second, columns may have text labels at the top, giving the name of the variable. Third, there may be header information before the data, which gives information about the dataset; less commonly, there can also be documentation at the end of the data. Finally, there may be missing data, and there are various ways to indicate missing data. Often it's done with a numeric code which is easily recognizable as being not real data. Sometimes, though, missing data is indicated by a character or group of characters (e.g. "MISSING", or "NA"). That can make problems with software that is expecting a number and tries to treat the entry as a number. All these things must be taken into account when reading in tabular data. The datasets to be analyzed in the Workbook sections of this book will all use numeric codes to indicate missing data, but one can always use a text editor to globally replace a text code with a numeric code.

This problem will be built around a tabular dataset of data from the Vostok ice core. The data set consists of four columns: the depth in the core, an estimate of the age of the ice, the deuterium composition of the ice relative to a standard (which is a measure of temperature at the time the ice was deposited), and a deuterium-based estimate of the past Antarctic temperature relative to the present. This file, and all the other ones you will need for this problem, are in the `Chapter1Data/iceCores` subdirectory of the Workbook datasets directory. First, open the file in a text editor and take a look at it to see what's there. Most text editors will not show the differences between tabs and spaces by default, though the better ones will have an option to make this difference visible. Read the header information, to get an idea of what the dataset means and where it came from.

Let's now try reading in some data from the Vostok ice core. The full file, which contains detailed documentation in a header plus column labels, with tab-delimited data following, is in the file you just looked at in the text editor. Lets start more simply, though. The file `vostokT.DataOnlyTabDelim.txt` contains only the data, with no header information whatever, in tab-delimited form. Read it in and take a look at some values. The description of the columns is given at the top of the file you looked at previously in your text editor. Next try reading in the data in space-delimited form, from file `vostokT.DataOnlySpaceDelim.txt`, to get an idea of how to handle that kind of data in your software. Then, try loading in tab-delimited data with column labels, from `vostokT.ColHeadTab.txt`. Finally, see if you can load in the data and column header part of the full file, `vostokT.txt`. Documentation is often provided at the top of tabular data, so if your software doesn't provide a procedure for skipping the header and extracting the data, you may need to use a text editor to do this by hand before loading data into your software for further analysis.

Once you have read in the data, make a graph of the temperature vs. ice age. Also, make a graph of the age vs. depth.

In addition to being able to read tabular data, you should be able to write tabular data to a file for later use. To practice this skill, take any of the Vostok ice core datasets you read in, and write a new text file which contains only the ice-age and temperature anomaly columns (the second and fourth columns of the original dataset). Read it back in and compare to the original data to make sure everything has worked properly.

Python tips: Data in a `Curve` object can be written to a text file using the `dump` method. In the courseware, the function `readTable` takes a filename as its argument, reads the data in the text file, and returns a `Curve` object containing all the data. This object can then be used for plotting, or for further data processing. The data must be in tabular form, with columns delimited by some consistent character (usually a tab). `readTable` is usually smart enough to strip off header information and spot where the data begins and identify column headings. The data set you read in typically has more information in it than you will want to plot on a single graph. To plot just selected columns, use the `extract` method of your `Curve` object. e.g. if `c` is your `Curve`, then `c.extract(['t','Z'])` will return a new `Curve` containing only the columns labeled `t` and `Z`.

Problem 1.1.3 *Arithmetic on columns*

One of the most basic forms of computation is the performance of arithmetic on rows or columns of tabular data. This kind of operation is very familiar from spreadsheets and commercial graphing software. All modern computer languages allow the performance of such operations on *arrays* or *vectors* without the need to write a loop dealing with each entry separately. An array or vector can be thought of as a row or column of data.

To exercise this skill, make up a column of data corresponding to pressure levels in an atmosphere. The pressure will be measured in millibars (mb), but the units are immaterial for this problem. The top of the atmosphere should start at 10 mb, and the data should extend to 1000 mb, which is a typical surface pressure. Next compute a column giving temperature on the *dry adiabat*, which you will learn about in Chapter **??**. The dry adiabat for Earth air is defined by the formula $T(p) = T_g \cdot (p/p_s)^{2/7}$, where T_g is the ground temperature and p_s is the surface pressure. Use $T_g = 300K$. Use the data in this column to make a third column which gives the corresponding temperature in degrees Fahrenheit (just hold your nose and pretend you are making a plot to be used in a television weather show).

To do a somewhat more interesting application of these skills, first read in the instrumental surface temperature record in the file `GISSTemp.txt`, located in the `GlobalWarming` subdirectory of the Workbook datasets directory for this chapter. These are estimates of global mean temperature. The first column gives the year, and the remaining columns give the temperature for each month, and for selected annual and seasonal averages. The temperatures are given as deviations from a baseline temperature, measured in hundredths of a degree C. Divide the annual average column (labeled "J-D") by 100 and add 14 C, to convert to actual temperature. Plot this curve and take a look at it to make sure it is reasonable (apart from the fact that it is going up at an alarming rate).

Now take the difference of the June-August column ("JJA") and the December-February column ("DJF"). These correspond to Northern Hemisphere summer, but remember that these are global means, so that Northern Summer is being averaged with Southern Winter. Global mean JJA still tends to be warmer than global mean DJF, because the seasonal cycle is strongest in the Northern hemisphere. However, note that the columns give the *deviation* of the temperature from the baseline average for these months, so the difference you computed gives the *change* in the strength of the global mean seasonal cycle, not it's absolute magnitude. Divide your result by 100 to convert the deviation to degrees. Do you notice any interannual variability in the strength of the seasonal cycle? What is the typical time scale? Do you notice an trend?

`Python tips:` The `Curve` object supports arithmetic on columns. Specifically, if `'x'` is a data column in the curve object `c`, then `c['x']` returns an array on which arithmetic can be done just as if it were a regular number. For example, `2*c['x']` is an array in which the entries are all doubled, `c['x']**4` returns an array consisting of the fourth powers of entries, etc. You can even do things like `c['T'] = c['T'] + 273.15` to convert a temperature from Celsius to Kelvin in-place, and can do similar things to create new columns that don't initially exist. You can compute the average of a column using constructions like `sum(c['T'])/len(c['T'])`. Many other array arithmetic operations can be done using the array functions available through `numpy` or its older version `Numeric`.

Problem 1.1.4 *Interpolation*

It commonly happens that one has data from observations or a model tabulated on a discrete set of points, but needs to be able to determine the value at an arbitrary point within the range covered by the data. The technique required to deal with this situation is called *interpolation.* An interpolation routine takes an array of values of the independent variable $[X_o, X_1, ..., X_n]$ and corresponding data values $[Y_o, Y_1, ..., Y_n]$, and on this basis provides a function $f(x)$ which estimates the value of the independent variable at an arbitrary value x of the dependent variable. In general, the list of independent variables should be sorted in monotonically increasing or decreasing order before carrying out interpolation.

The simplest form of interpolation is *linear interpolation*, where one determines the intermediate values by drawing a straight line connecting the neighboring data points. *Polynomial interpolation* is similar, except that one finds the m nearest points to x and uses them to determine an $((m-1)^{th}$ order polynomial that exactly goes through all the data points. m is usually taken to be a fairly small number, perhaps between 2 and 10. The case $m = 2$ corresponds to linear interpolation. There are more sophisticated interpolation techniques, which you can read about in *Numerical Recipes*, but polynomial interpolation will be adequate for our purposes.

Write a basic general purpose linear interpolation routine. Try it out on the data `X=[1,2,3,4]`, `Y=[1,8,27,64]`, subsampled from $y = x^3$, and plot the results at high resolution to make sure everything is working. Then, write or learn to use a polynomial interpolation routine. Do the same, and verify that the procedure recovers the cubic behavior exactly. Try subsampling the function $y = sin(x)$ with various numbers of points, and see how well polynomial interpolation of various orders reproduces the original function.

Next, read in the Vostok temperature dataset discussed in Problem 1.1.2 . The first column is depth, and this data set is already averaged down to a uniform 1 m depth interval for each data point. However, this layer depth corresponds to a longer time interval for old ice than for the young ice which has not been as stretched out; snow accumulation rate also affects the depth-time relation. The second column is the ice age, and this scale has non-uniform intervales. Suppose you need time series data on regular time grid with uniform 100 year intervals? Create such a series of deuterium composition (third column, a measure of the past temperature) by doing a polynomial interpolation using the ice age data as the independent variable array and the deuterium data as the dependent variable array. Try including different numbers of points in the polynomial interpolation to see how that affects the results.

Python tips: Polynomial interpolation from a table of values is implemented in the courseware by the function `polint`. This function evaluates a polynomial that goes through *all* the points in the table. In practice, this works best if the fit is done based just on a few points in the table nearest to the interpolation target. The full procedure, including identification of the n nearest points, is implemented by the `interp` class, which creates a function (strictly speaking, a callable object) that returns the interpolated value of the dependent variable given the desired dependent variable value as its argument.

Problem 1.1.5 *Root finding*

Find the solution of $x - \exp(-x) = 0$. First do it by the bisection method on the interval $[0, 1]$. Then do it using Newton's method.

Python tips: In the courseware, Newton's method is implemented via the class `newtSolve`, which creates an object that finds the zeros of a given function given an initial guess.

Problem 1.1.6 *Quadrature*

Evaluate the definite integral $\int_0^a \exp(-x^2)dx$ for $a = 2$ using the Trapezoidal Rule approximation. Then, used multiple Trapezoidal Rule evaluations together with Romberg extrapolation to obtain a more accurate result with less work. Then, use the numerical evaluation to show how the integral varies as a function of a. What happens at very large a?

Python tips: In the courseware, quadrature using the trapezoidal rule and Romberg extrapolation is implemented via the class `romberg`, which creates an object that carries out the integral. Note that if you want to try writing your own Romberg extrapolation quadrature routine, that the function `polint` can be used to do the needed polynomial extrapolation, as can the class `interp`.

Problem 1.1.7 *ODE integration in 1 dimension*

Numerically solve the logistic equation

$$\frac{dY}{dt} = aY \cdot (1 - Y) \tag{1.1}$$

subject to the initial condition $y(0) = .01$. Compare results using the midpoint method with results using Runge-Kutta, and discuss the dependence on the time step you use.

Python tips: In the courseware, the Runge-Kutta integration is implemented via the class `integrator`, which creates an object which carries out the integration. You should try implementing your own midpoint method as a loop in Python.

Problem 1.1.8 *ODE integration in 2 dimensions*

Numerically compute the solution of the system

$$\frac{dY}{dt} = -\frac{Z}{Y^2 + Z^2}, \frac{dZ}{dt} = \frac{Y}{Y^2 + Z^2} \tag{1.2}$$

subject to the initial condition $Y = 1$,$Z = 0$. Compare results for the midpoint and Runge-Kutta methods, and discuss the dependence on the time step you use.

Python tips: This can be done using the `integrator` class, which works for an arbitrary number of dependent variables.

1.1.2 Basic physics and chemistry

Problem 1.1.9 The acceleration of gravity on Earth is $10 m/s^2$, in round numbers. How much force (in N, i.e. *Newtons*) does a $1kg$ object exert when at rest? How much energy (in J, i.e. *Joules*) does it take to lift a $1kg$ object to a height of 100 m?

On the average about 200 J of solar energy falls on each square meter of Earth's surface each second – that's $200 W/m^2$ (1 W, or $Watt$, is 1 J/s). If all this energy could be converted to mechanical work, how many kg/s can be lifted to 100 m?

There are about 10,000 kg of air over each square meter of Earth's surface. How long would it take to lift all the Earth's air to an altitude of 200 km, if all the incoming solar energy could be used for this purpose? (You may assume the acceleration of gravity to be constant over this range of altitudes). Why doesn't this happen to Earth's atmosphere?

Problem 1.1.10 A small comet with a mass of $4 \cdot 10^9 kg$ strikes the Earth with a speed of 10 km/s. How much energy is released? Express the answer in Joules (J).

Problem 1.1.11 The surface gravitational acceleration on Titan is 1.35 m/s^2. The radius of Titan is 2575 km. Use Newton's law of gravitation to determine Titan's mass. (The gravitational constant G is $6.674 \cdot 10^{-11} m^3/kgs^2$).

Problem 1.1.12 Some aggressive space aliens halt the orbital velocity of the Moon, so that it falls like a rock upon the Earth. How much kinetic energy does the Moon have when it hits the surface? (*Hint* Use conservation of kinetic plus potential energy.). What is its speed at the time of impact, ignoring air resistance? Does the speed depend on the mass of the Moon?

Problem 1.1.13 An impactor with the mass of the Moon hits the Earth with a speed of 15 km/s. The object is made of silicate rock. Is the energy released enough to vaporize the object? Does your answer depend on the mass of the object? *Note:* It takes about $2 \cdot 10^7 J$ to vaporize 1 kg of silicate rock.

Problem 1.1.14 *Bolometric magnitude*

What is relevant for the climate of a planet is the power output of the planet's host star – its luminosity. What astronomers *observe* is the flux of energy from the star which reaches a telescope on Earth or in orbit. Suppose that the energy flux measured at the telescope is F. Astronomers generally transform the flux into *magnitudes*, defined by the expression

$$M = C - 2.5 \log_{10}(F) \tag{1.3}$$

where C is a constant chosen by convention which depends on the zero-magnitude reference. This magnitude is called the *apparent magnitude* because it corresponds to how bright the star seems as seen from Earth. The expression has its roots in the perception of relative brightness by human vision.

The magnitude defined above is called the apparent *bolometric magnitude* if it is based on the total flux over all wavelengths. In this case, the zero-magnitude convention is $C = -18.98$ if F measured in W/m^2. With this convention, the star Vega has approximately zero apparent magnitude. The Sun has an apparent bolometric magnitude of -26.82 as seen from Earth's mean orbit.

Assume that the star is at a distance d from the telescope and that its luminosity is $\mathcal{L}_{\circledast}$. The radiation from the star spreads out uniformly over a sphere of radius d, and there is no absorption by the intervening interstellar medium. Find an expression for $\mathcal{L}_{\circledast}$ in terms of the apparent bolometric magnitude and d/r_o, where r_o is the radius of the Earth's mean orbit. *Hint:* Use the apparent magnitude of the Sun, given above.

The red dwarf star Gliese 581,which hosts some interesting planets, has an apparent bolometric magnitude of 8.12, and is located at a distance of 6.27 parsecs. (1 parsec $= 3.086 \cdot 10^{16} m$) What is its luminosity relative to that of the Sun? If a planet were in orbit about Gliese 581, how close would it have to be to the star in order for the stellar flux incident upon the planet to equal the Solar flux incident upon Earth?

Absolute magnitude is the magnitude as seen from a standard distance of 10 parsecs. What is the absolute bolometric magnitude of the Sun? Of Gliese 581?

There's actually a bit more to learn about magnitudes. Astronomers typically report the magnitude as seen through various standard filters, for example "V-magnitudes" for yellowish light. These must be converted to bolometric magnitudes using a bolometric correction factor. This issue will be explored later, in Problem 3.1.11.

Problem 1.1.15 Luminosity is the net power output of a star, which we will measure in W. The proton process fuses four protons (hydrogen nucleii) into helium, yielding $4.17 \cdot 10^{-12} J$ of energy. Assuming constant luminosity, find a relation between the mass of the star and its lifetime on the main sequence, assuming the star to be initially hydrogen-dominated. How does your answer change if you use Eq. **??** to take into account the increase in luminosity over time? In this application of the formula, the parameter $t_\odot$ should assumed to represent half the star's lifetime on the main sequence, and the formula should be considered valid only out to times 2 $t_\odot$.

If T is the surface temperature of the star, then the radiation flux in W/m^2 out of the surface is $5.67 \cdot 10^{-8} T^4$. You will learn more about this formula in Chapter **??**. Use this to estimate the surface temperature of the star in terms of its radius. This also gives you the spectral class of the star, since cooler stars are redder. Then, if you have a relation between the mass of the star and its radius, you have a relation between lifetime and spectral class. Use this to show that the bluer stars are more short-lived, assuming that the density is independent of mass. A more accurate calculation would need to take into account the compressibility of the star's substance.

Problem 1.1.16 The Sun puts out energy at a rate of $3.84 \cdot 10^{26} W$. A resting human consumes (and puts out) energy at a rate of about $100W$. Look up the mass of the Sun, and determine which of these two bodies requires more energy production per unit of mass. Using the data given in Problem 1.1.15, determine how much mass of hydrogen would constitute a lifetime supply of food if humans could eat hydrogen and fuse it into helium?

Problem 1.1.17 In 2009, the world fossil-fuel emissions of carbon in the form of CO_2 was about 9 Gt (1 Gt = 1 $Gigatonne$ = 10^9 metric $tonne$ = $10^{12} kg$). If all of this carbon were turned into calcite ($CaCO_3$), how many Gt of calcite would you have? If it were spread evenly over the state of Kansas (surface area 213,000 km^2), how deep would the layer be? The density of calcite is about 2700 kg/m^3.

Problem 1.1.18 How many molecules of CO_2 are produced by the annual emission of $9Gt$ carbon? Earth's atmosphere contains about 10^{44} molecules in all, most of which are N_2 and O_2. If all the CO_2 from fossil fuel emissions stayed in the atmosphere, how much would the atmospheric CO_2 concentration increase each year? Express your answer in terms of the number of CO_2 molecules per million air molecules (i.e. ppm molar, also called $ppmv$).

1.1.3 Stable isotope calculations

Problem 1.1.19 Some ice that you dig up in Greenland has a $\delta^{18}O$ of -37 ‰. What is the ratio of ^{18}O to ^{16}O in this ice? Recent ice in Antarctica has δD = -438 permil. What is the ratio of D to H in this ice?

Problem 1.1.20 Assuming the mean δD for Antarctic ice to be -420 ‰, how much does the removal of 300 m depth of ocean water to form an Antarctic glacier shift the δD of the remaining ocean water?

Problem 1.1.21 Suppose that we write the equilibrium fractionation factor for phase 2 to phase 1 as $f_{1,2} = 1 + \epsilon$, where ϵ generally has small magnitude. Show that the shift in δ values between the two phases is

$$\delta_1 - \delta_2 = (1 + \delta_2)\epsilon \tag{1.4}$$

In the typical case, where δ_2 is small, this formula leads to the convenient approximation that the shift in δ between the phases is constant. This approximation breaks down if $|\delta_2|$ is not small. The precise statement of the physics at constant temperature is that the fractionation factor remains constant as mass is transferred from one reservoir to another. The approximation of a constant shift in δ is adequate for most common purposes, though.

Suppose that at $290K$ the $\delta^{18}O$ of water vapor in equilibrium with liquid having $\delta^{18}O = 0$ is -10 ‰. What is the value of ϵ?

Problem 1.1.22 A mass of liquid water with initial $\delta^{18}O$ of zero is brought into contact with dry air in a closed box. The mass evaporates until the air is saturated (i.e. holds as much water as it can). The isotopes are in equilibrium between liquid and vapor phases. The box is big enough that once equilibrium is reached, 1% of the mass of the water has evaporated. What is the $\delta^{18}O$ of the vapor at that point? What is the $\delta^{18}O$ of the remaining liquid? Then, new dry air is brought into the system, flushing out the moist air and the process is repeated, allowing another 1% of the remaining water to evaporate. What are the $\delta^{18}O$ values now? Continue in this fashion, and make a graph of the time series of the $\delta^{18}O$ of the vapor exhausted from the box and the $\delta^{18}O$ of the remaining water. You may assume that the entire experiment takes place at a fixed temperature of $290K$. (See Problem 1.1.21 for useful information.)

Problem 1.1.23 *A tale of two lakes*

A full lake is separated from an empty lake basin by a tall mountain range. The full lake initially is filled with water having $\delta^{18}O = 0$‰. A wind starts to blow from the full lake towards the empty basin, picking up water from the lake as it goes. The water that it picks up is in equilibrium with the lake water at all times. As the moisture-laden air blows up the mountain slope, moisture condenses out bit by bit. Each time moisture condenses out, it flows back into the originally full lake. The water that condenses is always in equilibrium with the vapor from which it condenses. As the moist air climbs the mountain, first 1% condenses out. Then another 1% of the remaining condenses out and so on. This happens 50 times by the time the top of the mountain is reached. What is the $\delta^{1}8O$ of the vapor that reaches the top of the mountain? You may ignore the effect of temperature on the fractionation factor, and assume it remains constant. (See Problem 1.1.21 for useful information.)

The air that crosses the top of the mountain mixes with some very frigid air, and all of the remaining vapor condenses out, falls into the initially empty lake and begins to fill it. Eventually the first lake will empty and all the water will have been transferred to the second lake. Describe the time evolution of the $\delta^{1}8O$ of each lake, shown as a function of the fraction of the initial lake volume that has been transferred from one lake to the other.

Now, suppose that instead of all the water being transferred, the second lake basin is shallow and there is a river channel connecting the top back to the first lake. When the second lake fills to one fourth the initial volume of the first, water flows back into the first lake at the same rate as water rains out into the smaller lake. The water running back mixes uniformly with the water in the big lake. What is the $\delta^{1}8O$ in each lake once this process reaches equilibrium? What would the $\delta^{1}8O$ be if the mountain were made taller, so that 75 precipitation steps of 1% each were made as the air crossed the mountain?

Say why this process is analogous to the isotopic effect of the transfer of sea water into an accumulating land glacier which eventually equilibrates by melting at a rate equal to the accumulation of new snow. State why making the mountain taller has an effect on the isotopic composition of the second lake which is analogous to the effect of a polar temperature decrease on the isotopic composition of the Antarctic or Greenland glacier.

Problem 1.1.24 During the $PETM$, a large quantity of isotopically light carbon was released into the atmosphere/ocean system. What does the change in measured $\delta^{13}C$ say about the amount of carbon released? This problem deals with that question in a simplified setting.

Suppose that 700 Gt of carbon was released into the atmosphere in the form of CO_2, from oxidation of organic carbon having $\delta^{13}C = -25‰$. Suppose all this CO_2 stays in the atmosphere. How much does the atmospheric $\delta^{13}C$ shift if the atmosphere initially contains 700 Gt of carbon in the form of CO_2 (roughly equal to present Earth levels)? Does the answer depend on what the initial atmospheric $\delta^{13}C$ is? What happens if the atmosphere initially contains 1400 Gt of carbon? What would happen if the carbon instead were released in the form of biogenic methane, with $\delta^{13}C = -50‰$, which was subsequently oxidized into CO_2 in the atmosphere?

The organic carbon release used in this problem is unrealistically low because on the real Earth, most of the CO_2 added to the atmosphere will be taken up by the ocean over a few thousand years, and the $\delta^{13}C$ in the atmospheric carbon will come into equilibrium with the $\delta^{13}C$ in the oceanic carbon. As a very crude attempt at incorporating this effect, assume that 7000 Gt was released, and that the ^{13}C is shared equally amongst a total atmosphere/ocean carbon pool which is about 100 times the mass of carbon in the atmosphere alone. Assume further that the $\delta^{13}C$ of carbonates deposited in the ocean mirrors the shift in the $\delta^{13}C$ of the total carbon pool. Compare your result to the 1.2 ‰lightening observed in $PETM$ carbonates. Note that one needs to know quite a bit of oceanic carbonate chemistry to do this problem properly, accounting for the isotopic exchanges of carbon amongst all the reservoirs in the system. Note further that the size of the carbon pool the atmosphere exchanges with is a quite separate question from how much of the added CO_2 remains in the atmosphere after various lengths of time.

Problem 1.1.25 Consider a planet with no photosynthesis. The planet has no ocean, though there is enough water around to keep the soil damp and sustain some forms of life. CO_2 outgassing from the interior creates a pure CO_2 atmosphere containing a mass M of carbon in the form of CO_2. Then outgassing ceases, and the atmosphere stops accumulating. The $\delta^{13}C$ of the carbon in the atmosphere is -6 ‰at this point. The surface mineralogy on this planet does not permit the formation of carbonates.

At this point, oxygenic photosynthesis evolves, and starts converting the atmosphere into O_2, which stays in the air, and organic carbon, which accumulates in a layer on the ground. The photosynthetic organisms fractionate carbon by -25 ‰*relative to whatever value the air has at the time they grow.* All their carbon comes from the air. In the first year, they convert 1% of the CO_2 into organic carbon, which accumulates in a layer. Then they die, and the next year a new crop grows, which converts 1% of the remaining CO_2. Graph the time series of the $\delta^{13}C$ in the air, resulting from this process. If, after a long time, one were to drill a core into the organic carbon layer, what would the profile of $\delta^{13}C$ vs. depth look like? You may assume that none of the carbon in the organic layer ever gets oxidized back into CO_2.

Now assume instead that the carbon cycle is in equilibrium between volcanic outgassing and organic burial. Each year, 1% of the carbon in the atmosphere is buried in the organic layer, and each year that same amount is replaced by outgassing at -6‰. Describe how the $\delta^{13}C$ in the organic layer and atmosphere behave in that case. Does the behavior depend on the equilibrium mass M of carbon in the atmosphere? What happens to the mass of O_2 in the atmosphere as time progresses? Why doesn't this happen on Earth?

Problem 1.1.26 The file `BenthicTropPacificB.txt` ,located in the `marineCores` subdirectory of the data directory for this chapter, contains $\delta^{13}C$ and $\delta^{18}O$ data from benthic forams, recovered from a tropical Pacific ocean sediment core. The data covers the past 5 million years. Read in the data. What is the typical variation in $\delta^{13}C$ over this time period? If the changes were entirely due to changes in organic carbon burial rate, which times would have high burial rate and which would have low burial rate? Try to estimate the implied changes in burial rate. (*Note:* In reality $\delta^{13}C$ is subject to a number of other influences, including the amount of CO_2 available in the ocean).

Now look at the $\delta^{18}O$ data. Convert the data to ice volume, assuming the entire variation is due to ice. Convert the data to temperature using the paleotemperature equation, assuming the entire variation is due to changes in the deep ocean temperature. Which limiting case do you think is closer to the truth?

Chapter 2

Thermodynamics and vertical structure

2.1 Workbook

Python Tips Many of the following problems require thermodynamic data such as specific heats, latent heats, etc. Most of the necessary data is given in the thermodynamic table for this chapter, and the rest can be found easily from standard print and online references. Python users can get the data directly from the `phys` courseware module, which defines gas objects for a number of common gases. Each gas object contains the data for the corresponding gas. For example `phys.N2.cp` is the specific heat (at standard temperature) for N_2. Type `help(phys)` and `help(gas)` for more documentation, or look directly at `phys.py`. The `phys` module also contains general constants such as the Boltzmann thermodynamic constant (`phys.k`) and many others, as well as a number of handy functions and routines which will be introduced below as needed. To use the `phys` module, type `import phys` from inside the Python interpreter.

2.1.1 Basic dry thermodynamics

Problem 2.1.1 A spherical spaceship with radius 10 meters is pressurized to an interior pressure of $1000mb$. The exterior pressure can be regarded as zero. What is the net force (in Newtons) exerted on the walls of the spaceship?

Problem 2.1.2 A hollow metal sphere of radius 1 meter is cut in half and then fitted back together with an airtight rubber gasket. All the air is then removed from the inside using a vacuum pump. The external air pressure is $700mb$. What is the total force (in Newtons) exerted by air on each hemisphere? What is the component of this force in the direction parallel to a diameter perpendicular to the plane along which the two hemispheres are joined? How much force would have to be exerted in order to pull the two hemispheres apart?

Problem 2.1.3 What is the density of Earth air at a temperature of 0 degrees C (centigrade) and pressure of $1000mb$? At the same pressure, what is the density of air in a hot air balloon in which the temperature has been increased to 50 degrees C? If the balloon is approximately spherical and has a radius of 3 meters, how much mass could it lift in Earth's gravity, if the ambient temperature is $0C$? (*Recall:* Archimedes' law states that the buoyancy force is equal to the weight of the fluid displaced. The opposing force on the balloon is the force of gravity on the air in the balloon itself, which much be subtracted from the buoyancy force.)

Problem 2.1.4 This is an elementary computer problem which exercises the skills needed to write and use a function.

Write a function `rho(p,T)` which returns the density of a gas given the pressure and the temperature. Check your function by recomputing the density of air at 0 degrees C and $1000mb$. The value of the universal gas constant can be hard-wired into the function, but you will need to find some way to specify the molecular weight of the gas in question. This can be hard-wired into the function (requiring you to edit the function when you want to change gases) or made into an additional argument of the function (requiring you to remember to pass a value which you probably aren't changing too often). A better solution is to put the data in a shared space that can be accessed by all functions. The particular way one does this depends on which computer language one is using.

Python Tips: Treat the molecular weight of the gas as a global, so that you can use the function for different gases by just re-defining the value of the global before calling the function.

Problem 2.1.5 A bicycle tire with a mass of $.1kg$ when empty, and a volume of $3liters$ is pumped up with Earth air to a pressure of $4bars$. It is pumped up slowly, so that its temperature remains at the ambient air temperature of $290K$. What is the mass of the tire after it has been pumped up? On Earth,what does the tire *weigh* (in $Newtons$)? What would the mass of the tire be if it were filled to the same pressure with CO_2 instead of air? With He?

Problem 2.1.6 Compute the density of CO_2 at the surface of Mars, where the pressure is $6mb$ and the temperature is $220K$. Compute the density of N_2 at the surface of Titan, where the pressure is $1.5bar$ and the temperature is $95K$. Compute the density of a pure CO_2 atmosphere at the surface of Venus, where the pressure is $92bars$ and the temperature is $737K$.

Problem 2.1.7 Assuming the Earth's atmosphere to contain 20% O_2 by count of molecules and 80% N_2, what is the mass mixing ratio of oxygen? What is the mass-specific concentration?

Problem 2.1.8 The air in a room with dimensions $3m$ by $20m$ by $20m$ has a CO_2 concentration of $300ppmv$. How many kilograms of *carbon* would have to be burned into the air (or exhaled by students) in order to double the CO_2 concentration in the room? If the O_2 concentration is initially 20% (molar), what is the concentration after the burning (or respiration) has taken place?

Problem 2.1.9 *Making the Earth's Oxygen*

Photosynthesis performs a kind of solar-powered electrolysis, taking water (H_2O) and carbon dioxide (CO_2) and using them to make organic matter– where the carbon and hydrogen and some of the oxygen go – and releasing the rest of the oxygen in the process. For the purposes of this problem, you can consider an organic molecule to be CH_2O, though real organics are typically more complex. The Earth's atmosphere contains about 20% O_2 (molar), and the oxygen can be considered well-mixed out to a great altitude.

Consider a patch of ocean of some surface area A. How deep a layer of water needs to be converted by photosynthesis to create the oxygen in the air above that patch? How many moles of organic matter will be made in the process? How many kilograms?

Problem 2.1.10 *First Law for a Non-Ideal Gas*

Beginning from the First Law in the form in Eq. **??**, show that for a gas with a general equation of state $\rho(p, T)$ the First Law can be re-written

$$\delta Q = c_p dT - \frac{1}{\rho^2}\frac{\partial \rho}{\partial p}|_T dp \tag{2.1}$$

and find an expression for c_p in terms of c_v. Show that when ρ is given by the ideal gas equation of state, your results reduce to the expressions derived in the text for the ideal case.

Problem 2.1.11 Suppose that sunlight heats a patch of the Earth's surface at a rate of 300 W/m^2, and that turbulence instantly distributes this heat over a column of boundary layer of depth $500m$. This means that 300 Joules of energy go into a column with base of 1 square meter and height of $500m$ each second. Assuming the boundary layer to be filled with Earth air with density $1kg/m^3$, at what rate does the temperature of the column increase, if there is no heat loss from the column? Re-do the calculation for the present Martian climate, with a pure CO_2 atmosphere having surface pressure of $6mb$, assuming the density to be constant over $500m$, the temperature to be constant at $240K$, and the solar radiation to be 150 W/m^2.

Problem 2.1.12 Use the Boltzmann constant to calculate the mean kinetic energy of a helium atom in a gas with a temperature of $300K$. What is the typical speed with which the atom moves? Recall that kinetic energy is $mv^2/2$, and that each direction of motion counts as a separate degree of freedom. (The mass of a helium atom is $6.642x10^{-27}$ kg).

Problem 2.1.13 *Temperature-dependent specific heat*

This is a computer problem involving temperature-dependent specific heats. To do the problem, you will need to know the following programming techniques: (1) Defining and evaluating a function; (2) Tabulating a set of values and either plotting them or writing them out for plotting with a separate program; (3) Summing up a set of values in a loop.

The *Shomate equation* is an empirical formula for the dependence of specific heat on temperature. It works well for a broad range of gases. The formula reads:

$$c_p = A + B*(T/1000.) + C*(T/1000.)^2 + D*(T/1000.)^3 + E*(T/1000.)^{-2} \tag{2.2}$$

where T is the temperature in Kelvins and $A, ...E$ are gas-dependent constants. Some coefficient sets are given in Table 2.1.

	A	B	C	D	E
N_2	931.857	293.529	-70.576	5.688	1.587
CO_2	568.122	1254.249	-765.713	180.645	-3.105
NH_3	1176.213	2927.717	-904.470	113.009	11.128

Table 2.1: Shomate coefficients for selected gases, valid from $300K$ to $1300K$ for CO_2 and NH_3, and $300K$ to $6000K$ for N_2

Write a function to compute `cp(T)`. Explore the behavior for the three gases, taking note of the degree to which the specific heat varies. Make a plot of the temperature dependence. From a physical standpoint, why do the specific heats increase with temperature? Measurements show that the specific heat of He is very nearly temperature independent. Why?

Compute the energy it would take to raise the temperature of $1kg$ of N_2 from $300K$ to $700K$, in steps of $.1K$ (assuming c_p to be constant within each step). Compare this to the value you would have gotten if c_p were assumed constant at its value in the middle of the temperature range. *Hint:* Write a loop to sum the energies in each step.

Python Tips: It is convenient to store the Shomate coefficients for each gas in a list,e.g.

```
N2_coeffs = [931.857,293.529,-70.576,5.688,1.587]
```

If you do this, you can set all the coefficients at once with a statement like `A,B,C,D,E = N2_coeffs`. A good trick is to define the lists as globals outside the function definition for `cp(T)`, and to have the function refer to the list of coefficients as, e.g. `ShomateCoeffs`. Then, to switch from one gas to another, all you have to do is set `ShomateCoeffs` to the list you want before you call `cp(T)`, as in `ShomateCoeffs = N2_coeffs`.

2.1.2 Potential temperature, dry adiabats and inhomogeneous atmospheres

Problem 2.1.14 The aforementioned bicycle tire with a volume of 3 liters is again pumped up with Earth air to a pressure of 4 bars. This time, it is pumped up so rapidly that there is not enough time for it to lose any heat to the surroundings. Assuming the ambient air has a temperature of $290K$, what is the temperature inside the tire immediately after it has been pumped up? What is the pressure after the tire has had time to cool down to the ambient temperature?

Problem 2.1.15 The atmosphere of Jupiter is predominantly H_2. A space probe determines that the temperature is $150K$ at a pressure of $10^5 Pa$. At what pressure does the temperature reach $300K$ if the atmosphere is well mixed and there is no condensation of any substance? Compare your results with the observed temperature profile for Jupiter given in the text.

Problem 2.1.16 The current atmosphere of Earth consists of about 20% O_2 by moles, and 80% N_2. Compute the dry adiabat (i.e. temperature vs. pressure with fixed θ) for the pre-biotic atmosphere without any oxygen, and compare with the dry adiabat for an atmosphere with 20% Oxygen and 80% Nitrogen.

Problem 2.1.17 You live on a planet with a pure CO_2 atmosphere with surface pressure of 1 bar. The acceleration of gravity is $10m/s^2$. Design a "nitrogen balloon" (filled with N_2) to lift a mass of $100kg$, under conditions when the low level temperature is $280K$.

Problem 2.1.18 A wind of pure N_2 with a temperature of $100K$ blows over a methane (CH_4) lake on Titan. The lake has a temperature T, and a thin layer of atmosphere in contact with the lake exchanges heat so that its temperature is also T. Enough methane evaporates from the lake to bring the molar concentration of methane in this layer to a value η. The surface pressure is $1.5bar$. Since we are assuming the atmosphere is pure N_2 above a thin layer in direct contact with the lake, the methane evaporation has essentially no impact on surface pressure. For what range of T and eta will the layer of atmosphere in contact with the lake be positively buoyant relative to the pure N_2 just above it?

Problem 2.1.19 Referring to the Venus temperature profile in Figure **??**, discuss the extent to which the Magellan observations of the temperature of the Venus atmosphere are consistent with a CO_2 dry adiabat. Extrapolate the temperature from the highest observed pressure to the $92bar$ surface of the planet, and compare the result to the observed surface temperature of $737K$. (Note that part of the mismatch is due to the assumption of constant c_p and the inaccuracy of the perfect gas law at high pressures)

$R/c_p \approx .2304$ for CO_2 gas.

Problem 2.1.20 In this computer problem, you will compute the dry adiabat $T(p)$ for an ideal gas whose specific heat depends on temperature, in accord with the Shomate equation (See Problem 2.1.13). In addition to basic skills such as defining functions and loops, you will need to know how to write programs that find approximate solutions to an ordinary differential equation of the form $dY/dx = f(x, Y)$

First, use the First Law of Thermodynamics to derive a differential equation for $d\ln T/d\ln p$ assuming $\delta Q = 0$. This defines the dry adiabat. Note that since c_p is a function of T, you can no longer treat it as a constant in doing the integral.

Write a program that tabulates approximate solutions to the differential equation. Note that your dependent variable is $Y \equiv \ln T$ whereas the right hand side of the differential equation involves T. This is not a problem, since you can write $T = \exp(\ln T)$. In writing your program, assume that $c_p(T)$ is defined by the Shomate equation.

Apply your program to obtain an improved approximation to the dry adiabat in a pure CO_2 Venusian atmosphere, which you originally computed in Problem 2.1.19. Start your computation at the ground ($p_s = 92 bars$) with the observed mean surface temperature of Venus ($737K$). Integrate up to the $100mb$ level, and compare the temperatures you get with those in the Magellan observations shown in Figure **??** of the text. Make a plot comparing your calculations with the dry adiabat obtained by keeping c_p constant at $820J/kg$.

2.1.3 Data Lab: Analysis of temperature soundings

Problem 2.1.21 *Analysis of Tropical Earth Soundings*

In this Data Lab we'll look at a set of balloon soundings of the atmosphere over the tropical Pacific, taken from shipboard during the CEPEX field campaign. The data is located in the subdirectory `cepex_sondes`. The files are named according to the date/time of release, in the form `yymmddhh.txt`.

To get started, let's read in a profile and take a look at it. First read the data from `93031512.txt` into your software. Note that the pressure is in millibars. The data files also contain relative humidity data, but we will not be using that for now. Make a plot of $T(p)$, and make sure it looks like the plot given in the text. To follow the usual conventions in a sounding plot, you should use a logarithmic pressure axis, and reverse the axis so that high pressure appears at the bottom.

We are now ready to take a more systematic look at tropopause height. The temperature minimum may be taken as a rough guide to the boundary of the troposphere, though it is known that there are some inadequacies in defining the tropopause this way. The general procedure is to look at the temperature array (or "column") `T` which has been read into your software, and find the index `i` which corresponds to the minimum temperature. Then the corresponding element `i` of the pressure array `p` is the tropopause pressure you want. The precise procedure for doing this depends on the language or software you are using. Try out the procedure on a few files in the data set, then write a function that takes the temperature and pressure array and returns the tropopause pressure.

Next, write a loop that loops over all the files in the data set and finds the tropopause pressure for each. (This may be easy or difficult, depending on what kind of programming language you are using). Summarize the results you find. How much does tropopause pressure vary among these soundings? All of the data in this dataset has been interpolated to a common set of pressure levels. This makes it easy to compute statistics over the sounding. Write a loop to compute the average temperature profile over the whole dataset. The loop will look rather like the one you used for analyzing tropopause height, but instead of just computing the tropopause pressure, the body of the loop will accumulate an array whose `i`'th element is the cumulative sum of the temperatures at the `i`'th pressure level. You would also keep track of the number of soundings read, then divide by this at the end to get the average. Try this out. Then improve your loop so it also computes the standard deviation of the temperature at each level about its mean at that level. At what heights does temperature vary the most? At what heights is it most weakly variable?

Next, pick a sounding, and use the ideal gas law to compute the density as a function of pressure. Remember that you need to convert the pressure from mb to Pascals when carrying out the calculation. Plot the computed density.

Now, write a function that computes the dry potential temperature profile $\theta(p)$ from data in a specified file. Look at a few plots. Which parts of the atmosphere are very statically stable? What part is closest to a dry adiabat? Process the entire dataset (or as much as you can), and examine the values of $\theta(600mb) - \theta(1000mb)$ and $\theta(200mb) - \theta(600mb)$. Discuss your results.

Python Tips:

Recall that you can use the function`readtable` in `ClimateUtilities` to read each sounding into a `Curve()` object. For example if `datapath` is the string containing the directory where the sounding data files are located, then

```
sounding = readTable(datapath+'93031512.txt')
sounding.YlogAxis = sounding.reverseY = sounding.switchXY =True
```

will read the data into a `Curve()` object called `sounding`. The second line sets the correct plotting options for a sounding.

There are a few functions available to Python which can help you find the temperature minimum and hence tropopause pressure. For any indexed object `data`, such as a list, the function `min` will find the minimum value in the list, and the list method `index` will find the index where that value occurs. Thus, if `T` is your temperature array, `i = T.index(min(T))` finds the index where the minimum occurs, then `p[i]` is the corresponding pressure. If you are using the `Numeric` or an equivalent array module, then `i = Numeric.argmin(T)` will directly find the index of the minimum temperature.

The `glob` function in the `glob` module, which is part of most standard Python distributions, can generate lists of filenames in a directory, which you can then loop over in doing your analysis. `glob.glob()` accepts wildcard characters in its argument, the most important of which is the asterisk. Thus, `glob.glob('*.txt')` returns a list of all the files in the current directory whose names end in ".txt". Try it out on a few of your directories, then try the following:

```
from glob import glob
from ClimateUtilities import *
datapath = <Your Path Here>
files = glob(datapath+'93*.txt') #Makes a list of files matching pattern
for file in files:
    sounding = readTable(file)
    i = Numeric.argmin(sounding['T']) #Finds index at which minimum T occurs
    print file,sounding['p'][i],T[i]
```

In doing manipulations with the data, you can make use of the fact that a `Curve()` object returns arrays on which arithmetic can be done without the need for a loop over elements. For example, to compute an average temperature, you first create an array of the correct length initialized to zero, using `Tbar = Numeric.zeros(len(sounding['T']),Numeric.Float)`. Then, inside the loop over files, you only need a statement like `Tbar = Tbar + c['T']`. Similarly, you can compute the dry potential temperature array from your temperature and pressure array using a single-line statement.

Problem 2.1.22 *Analysis of midlatitude and polar Earth soundings*

To see how the vertical structure of the atmosphere changes at mid and high latitudes, we turn to data from a global historical network of regular sounding stations. This dataset is rather sparse in the tropics, and even more so over the tropical oceans, which is why we used a special dataset to investigate the tropical structure. The dataset provided here is a very tiny subset of the vast CARDS sounding dataset, and is located in `CARDS_sondes`. The files are named according to the station number, date, and time. A list of stations and their locations is given in the file `stations.txt`.

Pick a few midlatitude stations, and look at the temperature profiles. How well defined is the tropopause? How does the tropopause height vary throughout the year? How do the values of $\Delta\theta/\Delta p$ in the lower and upper troposphere compare with those you found in the tropical case?

Now focus on Northern Hemisphere winter, and try to discover something about how tropopause height varies with latitude.

Finally, pick one of the Antarctic soundings, and take a look at the seasonal cycle of temperature. How high is the tropopause in Summer? Can a tropopause be identified in the dead of winter? What is the temperature structure near the ice surface? What do you think is happening there to cause that temperature structure?

Problem 2.1.23 *Analysis of planetary soundings*

First, we offer two very small datasets, which represent nearly all that is known about the vertical structure of the atmospheres of Jupiter and Venus. `Chapter2Data/GalileoProfiles/` contains one brave sounding of the Jovian atmosphere, conducted by the Galileo probe. Note that the pressure is reported in bars. Take a look at this sounding, and compare the temperature profile with the dry adiabat for various mixtures of H_2 and He, ranging from all hydrogen to all helium. From the appearance of Jupiter it is clear that the atmosphere is full of clouds, and hence condensation is widespread. Why do you think the dry adiabat works so well for this sounding?

The next small data set consists of a set of temperature profiles for Venus, obtained by examining the transmission of radio waves through the Venusian atmosphere. The first group of data is from a set of four probes dropped into the Venus atmosphere by the Pioneer Venus mission. These soundings are contained in `VenusPioneerProfiles`. The second group of data was taken by the Magellan probe, whose main mission was actually radar mapping of the surface of the planet. This data set only covers roughly the upper 7 bars of the atmosphere, whose surface pressure is $90 bars$. The soundings are contained in `Chapter2Data/VenusMagellanProfiles`.

Make plots of the soundings. Do you see any structure resembling a stratosphere? Do the temperatures differ much from one sounding to another? Compare selected Magellan soundings with the Pioneer Venus soundings.

In what part of the atmosphere do you think convection is occurring? Compare the temperature profile in this region with the dry adiabat for CO_2.

Now we turn to a bigger dataset, with more scope for exploration. The Mars Global Surveyor mission measured thousands and thousands of temperature profiles of Mars using the same radio occultation technique as the Magellan data set. The data extends nearly to the ground. The main limitation of this dataset is that, owing to orbital geometry, it covers a limited range of latitudes and times of day. In particular, one is limited essentially to late afternoon and night-time data. Hence, none of these soundings reflect times of most active convection through a strongly heated surface. The dataset is located in `Chapter2Data/MarsRadioOccProfiles`. What is provided here is only a tiny sample of the full dataset. The small subset of the data included in the Workbook datasets focus on the Southern Hemisphere, because the orbital geometry allows a better examination of the warmer part of the day than is the case for the Northern Hemisphere data. Some relatively warm Summer midlatitude evening soundings are given in the subdirectory `SummerSHevening`, while corresponding night-time and near-dawn soundings are in the `SummerSHnight` subdirectory. Some winter data in the Southern Hemisphere polar region is given in the `WinterSHPolarNight` subdirectory. The description line at the top of each file contains the geographic location of the sounding, and also local solar time and the angle the sun makes with the local vertical (the solar zenith angle). The information tag also contains the *subsolar latitude*, also known sometimes as the *latitude of the sun.* This is the latitude at which the sun is directly overhead at noon, and provides a useful measure of where we are in the seasonal cycle. For example, when the subsolar latitude is 0 degrees, we are at an equinox. When it is -20 degrees, it is near the Southern hemisphere summer solstice, and when it is +20 degrees it is near Northern hemisphere summer solstice. Note that not all of the soundings go all the way to the surface.

First take a look at the midlatitude early evening soundings. Do they generally contain a region where temperature decreases with height? Compute and analyze the profile of potential temperature for these soundings, assuming a pure CO_2 atmosphere with constant c_p. Are any of these soundings on the dry adiabat? What is the typical static stability (i.e. $d\theta/dp$)? The very high potential temperatures in the very upper part of the atmosphere may make it hard to see the structure in the lower portions, so you may want to try eliminating these from the graph. As an equivalent way of addressing the same question, compare the temperature profiles with the $T(p)$ corresponding to the dry adiabat based on the observed temperature nearest the ground.

Next take a look at the night-time midlatitude soundings. Where do these differ most from the early evening soundings? Compute and plot $\theta(p)$ for these soundings and discuss the results.

Now take a look at the polar night soundings. The main difference in the physical context of these soundings is that the midlatitude atmosphere has experienced the warming effect of the sun within the past day, whereas the polar night soundings have not been touched by sunlight for months. How does the low level temperature compare with that of the midlatitude soundings? How about the upper level temperature? Discuss the vertical profile. Are these soundings typically more or less statically stable than the midlatitude soundings?

Python Tips:

The description line at the top of a text data file is read into the returned `Curve()` object `c` as `c.description` when you read in the data using `readTable`).

You can use Python's array-cross section notation to subset the levels when computing potential temperature and creating a new curve object for plotting. For example if `c` is the `Curve()` object you read in from the dataset using `readTable`, `c['p'][0:10]` will give you the first 10 pressure levels, starting from the first pressure in the sounding (which is the highest pressure available, in this dataset).

2.1.4 The Hydrostatic Relation

Problem 2.1.24 Venus has a surface pressure of $92 bar$, and a surface gravity of $8.87 m/s^2$. 3.5% of the atmosphere (by mole fraction) consists of N_2. Compute the mass of N_2 per unit surface area of Venus, and compare with the corresponding number for Earth's atmosphere.

Problem 2.1.25 Suppose that the CO_2 concentration in the Earth's atmosphere is $300 ppmv$, and that it is well mixed. Compute the mass of *carbon* located over each square meter of the Earth's surface. Compute the total mass of carbon in the Earth's atmosphere. How does the carbon per unit area on Earth compare with the value for the present Martian atmosphere, consisting of pure CO_2 with a surface pressure of about $6mb$?

Problem 2.1.26 Titan has a mostly Nitrogen atmosphere with a surface pressure of about $1.5 bars$. The gravitational acceleration at the surface of Titan is $1.35m/s^2$. Titan's radius is about $2600km$. How much mass of atmosphere is there above each square meter of surface? How does this compare to the mass of air over each square meter of Earth's surface? What is the total mass of Titan's atmosphere?

Problem 2.1.27 The temperature of Titan's troposphere, which contains most of the mass of the atmosphere, is about $100K$. The atmosphere is mostly N_2. Estimate the density scale height of Titan's atmosphere and compare it to Titan's radius. How does the situation on Titan differ from that of Earth? For which body is the atmosphere more accurately described as a "thin spherical shell"?

Problem 2.1.28 *An atmosphere for the Moon*

At some time in the future the surface of the Earth becomes uninhabitable, and the few remaining residents decide to move to the Moon. To make life more congenial, they decide to give the Moon an N_2/O_2 atmosphere with a surface pressure of 1 bar. They also add in enough carbon dioxide to bring the mean surface temperature up to $280K$. How much total mass needs to be brought in to create this atmosphere? If the mass comes in the form of comets, each $10km$ in radius, how many are needed? Estimate the scale height of the atmosphere, and compare it to the radius of the Moon.

Python Tips: The radius of the moon is `planets.Moon.a`, and the acceleration of gravity at the surface of the Moon is `planets.Moon.g`.

Problem 2.1.29 Use the hydrostatic relation show that $dT/dz = -g/c_p$ assuming the dry potential temperature to be constant. Do this first by differentiating the dry adiabatic temperature profile $T(p)$ and using the chain rule and hydrostatic relation. Then show that you get the same answer directly by assuming the dry static energy to be independent of height.

The quantity g/c_p is called the *dry adiabatic lapse rate.* Compute the typical value for dry air on Earth, CO_2 on Mars, CO_2 on Venus, and N_2 on Titan (all based on surface gravity).

Problem 2.1.30 In this computer problem, you will find numerical solutions to the hydrostatic equation for an ideal gas atmosphere with general $T(p)$ (alternately $T(z)$). The solution is done in two different ways. The techniques you will need are: (1) Numerical quadrature (i.e. evaluation of a definite integral); and (2) Numerical solution of an ordinary differential equation.

As a prelude, first integrate the hydrostatic relation analytically, to find $p(z)$ for an atmosphere whose temperature profile is given by the ideal gas dry adiabat, $T(p) = T_s(p/p_s)^{R/c_p}$. From the answer to Problem 2.1.29, this is equivalent to $T(z) = T_s - (g/c_p)z$, where g is the acceleration of gravity. This solution will serve as a check on your numerical solution.

One approach to evaluating the hydrostatic relation is to use Eq. **??**, which gives you $p(z)$ directly once you know $\bar{T}(z)$. The temperature $\bar{T}$ can be found for any z by carrying out the definite integral given in Eq. **??**. This works as long as you know T as a function of z. Write a program to find $p(z)$ using this method. To keep your program general, let the temperature profile be specified by an arbitrary function `T(z)`. Apply your program to the dry adiabat for Earth air, with surface pressure $1bar$ and surface temperature $300K$. Compare with the analytical result.

The above method doesn't work if you know $T(p)$ rather than $T(z)$. Since it is easy to measure pressure and temperature but relatively hard to measure altitude, balloon and planetary probe soundings often provide $T(p)$. There is an easy fix for this problem, since Eq. **??** can be re-written as $dz/d\ln p = -RT(p)/g$, whence one can get $z(\ln p)$ by carrying out a definite integral with respect to $\ln p$. Write a program to do this, and test it on the dry adiabat. Note that if you want a graph of $p(z)$, you can just make a graph of $z(p)$ using this routine, and switch the axes (or equivalently, turn it on its side).

Challenge: Note that if you want $p(z)$ for a whole range of altitudes, it is inefficient to do the calculation from scratch for each new value of z, using either of the above methods. By dividing up the definite integral for $\bar{T}$ into integrals over sub-intervals, write a program that efficiently computes $p(z_2)$ given $p(z_1)$ where $z_1 < z_2$. Do not assume that the temperature is nearly constant between z_1 and z_2.

The second approach is to solve the hydrostatic relation by numerically integrating the ordinary differential equation in the form given by Eq. **??**. This method can actually be faster, since it doesn't require the evaluation of exponentials, and has the virtue of very easily giving results for a whole range of altitudes, without having to re-do the calculation from scratch for each new altitude. Since T is a function of p, Eq. **??** is an ODE of the form $dY/dx = f(x,Y)$ with $Y \equiv p$ and $x \equiv z$. Write a program to solve the program in this form. Apply it to the dry adiabatic profile defined above, and check against the analytic result.

2.1.5 Latent Heat and Clausius-Clapeyron

Problem 2.1.31 A spherical comet $1km$ in radius and with a density twice that of water hits the ocean with a velocity of 20 km/s. If all the energy is converted to heat and used to vaporize water, how many kilograms of steam result?

Problem 2.1.32 Consider a glacier of water ice that is $1km$ thick and covers an area of $10^8 km^2$ (about a fifth of the area of the Earth). Suppose that the ice absorbs $50W/m^2$ of solar radiation, and all of this is used to either sublimate (into gas) or melt (into water) the ice. How long does it take for the glacier to disappear in each of these cases? Give your answer in years. The density of ice is about 930 kg/m^3 . What determines whether sublimation or melting occurs?

Problem 2.1.33 Titan receives somewhat less than $5W/m^2$ of sunlight at its surface. If all of this energy were used to evaporate methane, about how long would it take to evaporate a lake of methane with a depth of 1 meter?

Python Tips: The latent heat of evaporation of methane is given in `phys.CH4.L_vaporization`.

Problem 2.1.34 This computer exercise requires that you learn how to write a function in a programming language. It applies this skill to evaluating the Clausius-Clapeyron relation.

The simplified constant L form of the Clausius-Clapeyron relation is adequate for many purposes. To evaluate it, you need to specify a reference temperature and vapor pressure, the latent heat of the phase transition under consideration, and the gas constant (or equivalently the molecular weight). Using your favorite programming language, write a function `psat(T)` which computes the saturation vapor pressure as a function of temperature. The constants needed to evaluate the function can be treated as global constants, hard-wired into the function, or added to the argument list.

Put in the constants appropriate to CO_2 gas in equilibrium with a solid. It is convenient to use the triple point pressure and temperature for the reference pressure and temperature. Use the function to estimate how low temperature would have to be for CO_2 to condense at the Martian surface, assuming the $7mb$ surface pressure typical of the present planet.

Python Tips: Commonly, one makes a function because one wants to evaluate it many times for the same gas but different values of T. If you add the thermodynamic constants to the argument list, you are stuck typing in a long argument list when all you really wanted to change is T. Making the constants into globals solves this problem, but then makes it hard to deal with saturation vapor pressure functions for several different gases in the same program. `phys` provides a better way to deal with the problem, making use of the fact that objects can be made callable. Namely, it provides the class `satvps_function` which takes the needed gas parameters as arguments and creates a function-like object that stores the constants it needs. The triple point temperature and pressure provide a convenient set of base conditions for use in the formula. Thus, to create a constant-L saturation vapor pressure function for CO_2 sublimation, one would write

```
gas = phys.CO2 # get the gas object you want
pCO2 = phys.satvps_function(gas.TriplePointT,
  gas.TriplePointP,gas.MolecularWeight, gas.L_sublimation)
```

This creates a callable object `pCO2`, whereafter you can evaluate the vapor pressure by simply writing `pCO2(T)`. This a simple but very useful technique. Take a look at the class definition in `phys` to see how it is done, since you may want to use the technique yourself sometime.

Actually, it's pretty tedious to have to enter all the thermodynamic constants in the argument list. Why not just give all the constants at once in the form of a `gas` object? In fact, through the power of *polymorphism*, the `satvps_function` class recognizes what kind of thing it's given as an argument, and acts accordingly. Therefore, you could get the same result as above using `phys.satvps_function(phys.CO2)`. By default, this will create a function that uses the latent heat of sublimation for temperatures below the triple point, and the latent heat of vaporization for temperatures above the triple point. But what if you want to force the function to use one latent heat or the other regardless of temperature? For this purpose, the class allows an optional second argument. If it's value is the string `'ice'`, the latent heat of sublimation is assumed. If it is `'liquid'` the latent heat of vaporization is assumed.

Problem 2.1.35 Titan has a surface temperature of about $95K$. Suppose that there is a lake of liquid methane ($CH4$) on the surface, and that the air just above the lake is in equilibrium with the lake, and therefore saturated with respect to methane. Use the simplified form of Clausius-Clapeyron to determine the partial pressure of methane in the air near the surface. What would the partial pressure be if the temperature rose to $120K$? Based on an N_2 partial pressure of $1.5bar$ at the surface, what would the molar mixing ratio of methane in the atmosphere be? What would the mass mixing ratio, the molar concentration and the mass specific concentration be? What would the partial pressure and molar mixing ratio be if the temperature rose to $120K$? What would the mass mixing ratio, the molar concentration and mass specific concentration be?

Problem 2.1.36 Transition from liquid to vapor (evaporation) can happen at any temperature. It doesn't require a threshold temperature to be exceeded before it can happen. What then is the "boiling point," which certainly represents a kind of vapor-liquid transition? In fact, boiling is not much different from evaporation. What is special about boiling is that the saturation vapor pressure becomes greater than the `total` atmospheric pressure at the liquid surface. This means that the vapor pressure can push aside fluid and form bubbles in the interior of the fluid, which can then escape by rising. It is a much faster process than evaporation, because the transition can happen throughout the fluid, and not just at the gas-liquid interface.

Compute the boiling point of water at an altitude where the pressure is $300mb$.

Suppose a river with a temperature of $300K$ erupted from the interior of present Mars, where the surface pressure is about $7mb$. Would the river boil? How high would the surface pressure have to be to prevent boiling at this temperature?

Now, think of Glurg the Titanian, who would like to boil up liquid methane to make his tea. The surface pressure of Titan is about 2 bars (mostly Nitrogen). How hot does his stove have to get?

Problem 2.1.37 *Water on the Hadean Earth*

Ancient zircon crystals tell us that liquid water existed somewhere on the surface of the Earth as early as 4.2 billion years ago. What does this tell us about the surface temperature? Does it imply that the temperatures had to be below $100C$ (the "boiling point")? To answer this question, compute the saturation partial pressure of water vapor at an ocean surface as a function of surface temperature. Do this three ways. First, use the exponential form of the Clausius-Clapeyron relation with constant latent heat, based on triple point data. Next, use the exponential form based on "boiling point" data (i.e. that the vapor pressure is $1bar$ at $373.15K$. Finally, use the empirical Antoine equation:

$$p_{sat}(T) = 10^{A-B/(T+C)} \tag{2.3}$$

where A,B and C are empirical coefficients determined by fits to experimental data. In the range $373-647K$, the coefficients are $(A,B,C) = (5.2594, 1810.94, -28.665)$, if the pressure is given in *bars* and temperature in K. This fit gets the critical point pressure slightly wrong, but it is adequate for the purposes of the present problem.

Using your results, answer the following. At $100C$ ($373.15K$) what is the partial pressure of water vapor at the surface? What fraction of the molecules of the atmosphere are water at the surface (assuming that the non-condensing part of the atmosphere has partial pressure of $1bar$)? Does the ocean boil under these conditions? What if the surface temperature is $200C$ instead? At this temperature, what proportion of the mass of water on the planet is in vapor form in the atmosphere, assuming the total water mass to be the same as that of the present ocean ($1.4 \cdot 10^{21}kg$)? How hot would the surface have to be in order for all the inventory of water to go into the atmosphere? Specifically, show that at the critical point temperature for water, almost all of the Earth's oceanic inventory of water has gone into the atmosphere. Do you think that it is a coincidence that the equivalent pressure of the Earth's ocean is close to the critical point pressure?

Hint: At temperatures of $400K$ and above, it is a reasonable approximation that water vapor dominates the mass of the atmosphere, so that by the Hydrostatic Law, the mass of water vapor in the atmosphere can be approximated by $4\pi a^2 \cdot p_{sat}/g$, where p_{sat} is the saturation water vapor pressure at the surface (in Pa) and a is the radius of the Earth. Problem 2.1.54 will show you how to accurately compute the total mass of water in cooler conditions when water vapor does not dominate the atmosphere.

Remark: Although you will find that the existence of liquid water does not itself limit the surface temperature to below $100C$, the oxygen isotopic content of the zircons does tend to argue for moderate surface temperatures.

Problem 2.1.38 Suppose that the latent heat of vaporization varies linearly with temperature, according to the law $L = a - bT$ with a and b positive, up to the temperature where latent heat vanishes. Use the Clausius-Clapeyron relation to find an expression for $p_{sat}(T)$, given its value as some T_0. Make a sketch illustrating how the saturation vapor pressure differs from the constant-L case.

Problem 2.1.39 In this computer problem you will implement functions which provide empirical fits to the vapor pressure for water, taking into account the variations of latent heat with temperature.

It has been found that the function $satvpw(T)$ defined below provides an accurate fit to the measured vapor pressure over liquid water, for temperatures between the freezing point and $400K$.

$$satvpw(T) = e_o \cdot 10^{a+b+c+d+e} \tag{2.4}$$

where

$$\begin{aligned} a = -7.90298 \cdot (T_o/T - 1), b = 5.02808 \log_{10}(T_o/T),\\ c = -1.3816 \cdot 10^{-7} \cdot 10.^{11.344(1-T/T_o)-1},\\ d = 8.1328 \cdot 10^{-3} \cdot 10.^{-3.49149(T_o/T-1)-1} \end{aligned} \tag{2.5}$$

where $T_o = 373.15K$ and $e_o = 101324.6Pa$. For ice, the corresponding function $satvpi(T)$ has the same form but with $T_o = 273.15K$, $e_o = 610.71Pa$,

$$a = -9.09718(T_o/T - 1), b = -3.56654 \log_{10}(T_o/T), c = 0.876793(1 - T/T_o) \tag{2.6}$$

and $d = 0$. The fit for ice is valid from the freezing point down to about $100K$.

As it turns out, there's a further complication: in real atmospheres, liquid water can exist in a supercooled state well below the triple point temperature. Typically, the mix of supercooled water and ice shifts from all water to all ice as the temperature is reduced below the triple point. The following function (taken from a typical climate model) uses the empirical function for ice for temperatures below -20C, the empirical function for liquid for temperatures above 0C, and shades linearly between ice and water saturation as temperature is increased from -20C to 0C, specifically:

$$satvpg(T) = 0.05(273.15 - T) * satvpi(T) + 0.05(T - 253.15) * satvpw(T) \tag{2.7}$$

where $satvpi$ and $satvpw$ are the vapor pressures over ice and liquid water,defined previously.

Write functions to compute $satvpw(T)$, $satvpi(T)$ and $satvpg(T)$. Compare $satvpw(T)$ with the idealized saturation vapor pressure function for water based on a constant latent heat of vaporization, using the triple point temperature and pressure as the base values. Can you see the difference between the two when you plot them both with a logarithmic pressure axis? To get a more accurate view of the error, compute a table of $(satvp1(T) - satvpw(T))/satvpw(T)$, where $satvp1$ is the idealized function. Try using the boiling point pressure and temperature ($1bar$ at $373.15K$) as the reference value for $satvp1$ and see how the comparison differs.

Compare the empirical and idealized vapor pressures for ice in the range $150K$ to $273K$.

Compare $satvpw(T)$ with a simplified function consisting of the constant L function for ice below the triple point and the constant L function for liquid water above the triple point. In the simplified function, you should use the triple point temperature and pressure to provide the reference value. Do the comparison for the range $250K$ to $290K$.

Python Tips: The functions described above are provided for you as `phys.satvpw`,`phys.satvpi` and `phys.satvpg`

Problem 2.1.40 Suppose that you are given a function $e(T)$ which gives the saturation vapor pressure of a substance as a function of temperature, and you need to find the latent heat of the phase transition, as a function of T. This situation might arise if the vapor pressure function you are using arises from an empirical fit to data. This is the case for the function $satvpg(T)$ in Problem 2.1.39.

Use the Clausius-Clapeyron relation to derive a formula for L in terms of $d\ln e/dT$. You can approximate the derivative using the formula

$$\frac{d\ln e}{dT} \approx \frac{\ln e(T+dT) - \ln e(T-dT)}{2dT} \tag{2.8}$$

where dT is some suitably small number. Writing the derivative function this way allows you to pass any saturation vapor pressure function to the procedure, and get an answer without recoding. Now write a function using the approximate derivative which returns the latent heat for any argument T. It's a good idea to make dT an argument of the function, so that you can try out different values easily. Note that in most computer languages the function name `e` can also be passed to your latent heat function as an argument. If you do this, you can use your latent heat function for several different choices of $e(T)$ without recoding. The function needs the gas constant R for the material in question. You could add the gas constant to the argument list of the function, but it makes more sense to leave it as a global to be set in your main program before you use the function.

First, check your latent heat function by passing it a saturation vapor pressure function made using the constant L approximation, and make sure that you get the right value of latent heat back. What happens if you make dT too small? If you make it too large?

Now apply your procedure to *satvpg*. Plot the results in the range from $200K$ to $400K$, and discuss. Compare the computed latent heat with the computed latent heats for *satvpw* and *satvpi*. Why do you think the effective latent heat for *satvpg* has a spike near -20C?

Problem 2.1.41 Apply the function you wrote in Problem 2.1.40 to determine the implied latent heat as a function of temperature in the Antoine vapor pressure formula used in Problem 2.1.37.

Problem 2.1.42 This problem is inspired by Fritz Leiber's story, "A Pail of Air." The Earth has a close encounter with a large interplanetary object, and is flung off into the cold night of space far from the Sun. As the Earth gradually cools down the atmosphere begins to condense, until essentially all of the atmosphere is deposited in a condensed layer at the surface of the planet. For the purposes of this problem, you may assume that dry Earth air consists of a mixture of 80% N_2 and just under 20 % O_2 (by mole fraction), with the balance consisting of CO_2 with a mole fraction of 380$ppmv$. To keep things simple, we'll ignore atmospheric water vapor, which would quickly snow out and add to the existing surface layer of water ice and snow. The Earth's initial surface pressure is $10^5 Pa$, and the acceleration of gravity is $9.8 m/s^2$.

Describe the sequence of events that occurs as the atmosphere cools. Will the atmosphere condense into solid or liquid forms? Will all the components of the atmosphere be mixed together at the surface, or will they occur in layers? Why? How thick would the layer of each substance be, and in what order would the layers occur? For the purposes of this problem, you may assume that any snow (of any substance) that forms becomes so compacted that the density is the same as that of the pure solid.

Problem 2.1.43 *Water content of Earth's atmosphere*

Consider an atmosphere with the temperature profile:

$$T(p) = \begin{cases} T_s \cdot (\frac{p}{p_s})^a & \text{for } p > p_t, \\ T_s \cdot (\frac{p_t}{p_s})^a & \text{otherwise.} \end{cases} \tag{2.9}$$

This provides an adequate model of the tropical temperature profile with $p_t = 100mb$ and $a = .176$. Find the profile of the saturation specific humidity for water,q_s, associated with this profile. What happens in the isothermal stratosphere if you insist that the water vapor remain saturated? Suppose now that there is no water in the stratosphere ($p > p_t$). From the definition of specific humidity, the total mass of water per square meter in a column of the atmosphere is then:

$$m_w = \int_{p_s}^{p_t} q_s \frac{dp}{g} \tag{2.10}$$

Compute this integral approximately, by dividing the column up into many thin layers and summing the water in each. Discuss the dependence on surface temperature. The *liquid equivalent depth* is the depth of the layer of liquid water that would be produced if all the vapor were condensed out of the column and spread uniformly over its base. It is m_w/ρ_{liq}, where ρ_{liq} is the density of liquid water (about 1000 kg/m^3). What is the equivalent liquid depth of water in a saturated tropical atmosphere?

Problem 2.1.44 Examine some of the polar winter soundings from the Mars Global Surveyor Radio Occultation dataset introduced in Problem 2.1.23, and discuss whether there are any parts of the atmosphere where condensation of CO_2 into CO_2 ice is likely to have occurred. To do this, you need to use the Clausius-Clapeyron relation to compute the saturation vapor pressure at each level in the sounding, using the temperature data, and then compare the saturation pressure with the actual pressure. Alternately, you can solve the simplified Clausius-Clapeyron formula for $T(p)$ and compare this directly with the temperature sounding.

Problem 2.1.45 According to some interpretations of geologic data from the Neoproterozoic (about 600 million years ago), the Earth went through one or more episodes of global glaciation. The phenomenon is known as ”Snowball Earth.” The general thinking has it that the Earth would exit from a Snowball state after sufficient CO_2 has built up in the atmosphere to warm the planet to the point where it deglaciates.

In the Snowball state, the planet becomes very cold, because an ice surface reflects a great deal of solar radiation. Simulations show that the surface temperature at the Winter pole can drop to $160K$, though it warms to about $240K$ in the balmy polar summer. Under these circumstances, how high would the CO_2 partial pressure have to rise before CO_2 condensation sets in at the winter polar surface? Based on a 1 bar surface pressure for the non-CO_2 part of the air what would the molar mixing ratio and molar concentration of CO_2 have to be to cause condensation? How do your answers change if the surface temperature is increased to $200K$? What do you think condensation would do to the global CO_2 concentration?

2.1.6 Moist Adiabats

Problem 2.1.46 *Springtime for Europa*

Something is about to happen. Something wonderful. To promote life on Jupiter's moon Europa, which currently is composed of a liquid water ocean covered by a very thick water ice crust, the alien race which built Tycho Magnetic Anomaly 1 ignites thermonuclear fusion on Jupiter, heating Europa to the point that its icy crust melts, leaving it with a globally ocean covered surface having a temperature of $280K$. Water vapor is the only source of atmosphere for this planet. Describe what the atmosphere would be like, and calculate $T(p)$ for this atmosphere. Give a rough estimate of the depth (in km) of the layer containing most of the mass of the atmosphere. (The gravitational acceleration at Europa's surface is $1.3m/s^2$).

Problem 2.1.47 *Martian polar winter temperature structure*

During the winter near each of the Martian poles, CO_2 condenses out from the atmosphere to form a layer of CO_2 snow which accumulates at the surface. As springtime approaches, this snow layer sublimates into the atmosphere until it is completely gone, leaving only the underlying water ice behind. What is the surface pressure when the surface temperature is $145K$, $150K$, $151K$ and $152K$, assuming that some CO_2 ice remains at the surface in these conditions? Plot the temperature profile $T(p)$ you expect for these cases, and justify your reasoning. In each case, show results for p ranging down to $\frac{1}{1000}$ of the surface pressure. Do you think any CO_2 snow is likely to be left at the surface of Mars when the polar temperature reaches $160K$?

For the purposes of this problem, you may assume that Mars has a pure CO_2 atmosphere.

Hint: To make it easier to compare the profiles, you may want to use $-\ln(p/p_s)$ as your vertical coordinate.

Problem 2.1.48 Consider a planet with a methane-ammonia atmosphere, both of which are assumed in saturation so that lifting will case both methane and ammonia to condense. Make a table and plot of the temperature vs. pressure for this atmosphere, and of the mixing ratio of methane to ammonia as a function of pressure.

Hint: The partial pressures of the two individual components are given simply by their respective Clausius-Clapeyron relations. How do you then get the total pressure from the partial pressures? Note that finding an expression for $T(p)$ is difficult, but computing $p(T)$ is easy. You can use the later to make graphs and tables just as easily as the former.

Problem 2.1.49 Using the formula for the slope of the saturated moist adiabat for a condensable/noncondensable mixture, compute $d\ln(T)/d\ln(p_a)$ for a mixture of water vapor and Earth air at a pressure of $1000mb$ and (a) a temperature of $300K$, (b) a temperature of $250K$. List the values of the individual terms in the numerator and denominator, so that you can get a feel for which are big and which are small. Compare the values with the slope of the dry adiabat. Note that $p_a \approx p$ in this temperature range, which simplifies the calculation.

Problem 2.1.50 Consider an atmosphere on the saturated one-component moist adiabat $T_{ad}(p)$ given by Eq. **??**. Suppose that a parcel of atmosphere is lifted from a pressure p to a slightly lower pressure $p - \delta p$ (with δp ¿0). Suppose that the parcel is lifted without condensation, i.e. along the dry adiabat. By comparing the resulting temperature with $T_{ad}(p - \delta p)$, show that condensation occurs on ascent provided the slope $d\ln T_{ad}/d\ln p$ is less than the dry adiabatic slope R/c_p. Using Clausius-Clapeyron to compute the moist adiabatic slope, show that this criterion is equivalent to $c_pT/L < 1$. Show that if the criterion is not satisfied, then condensation occurs on descent instead.

Another way to state the criterion is to define the characteristic temperature L/c_p, and to say that condensation occurs on ascent when T is below the critical temperature. Compute L/c_p for CO_2, H_2O, N_2, CH_4 and NH_3, using both the latent heat of sublimation and the latent heat of vaporization in each case. For the purposes of this problem you may use the latent heats and specific heats given in Table **??**. Show that in each of these cases the criterion for condensation on ascent is automatically satisfied as long as the temperature is below the critical point temperature. Note that to determine what happens as the critical point is approached, one must use a more general form of Clausius-Clapeyron and take into account the temperature variations of L and c_p, as well as the departures from the ideal gas equation of state.

Problem 2.1.51 In this problem you will compare the moist adiabatic slope with the dry adiabatic slope for an arbitrary mixture of a saturated condensable gas with a noncondensable gas. Though the calculation can be done manually, it is much easier to write a small program to carry out the procedure outlined below, given temperature and gas properties as input. For the purposes of this problem you may assume the latent heat to be independent of temperature.

Divide up the problem as follows. First, using the saturation vapor pressure, compute R/c_p for the mixture using the appropriately weighted gas constants and specific heats for the two components. This is the dry adiabatic slope. Next, using Eq. **??**, compute $d\ln T/d\ln p_a$. Finally, to convert this to $d\ln T/d\ln p$ you need to multiply by $d\ln p_a/d\ln p$, which is equal to $(p/p_a)(dp_a/dp)$. Compute this quantity by noting $p = p_a + p_c$, $d(p_a + p_c)/dp_a = 1 + dp_c/dp_a$, and $dp_c/dp_a = (dp_c/dT)/(dT/dp_a)$ You can directly evaluate (dp_c/dT) using Clausius-Clapeyron, since the condensable substance is assumed saturated. You can evaluate (dT/dp_a) using the value of $d\ln T/d\ln p_a$ which you computed already.

Carrying out this procedure allows you to evaluate the moist adiabatic slope with a minimum of tedious algebra. For a range of temperatures extending from temperatures where the condensable is dilute to temperatures where the condensable is dominant, compare the moist adiabatic slope to the dry adiabatic slope for each of the following mixtures: (1) Condensable H_2O in N_2, (2) Condensable H_2O in H_2, (3) Condensable CH_4 in H_2, (4) Condensable CO_2 in N_2, (5) Condensable CH_4 in N_2.

Problem 2.1.52 *Numerical computation of dilute moist adiabat*

For the dilute case, in which the mixing ratio of the condensable substance is small, the slope of the moist adiabat is given approximately by $d\ln T/d\ln p = (R_a/c_{p,a})(1 + A)/(1 + A \cdot B)$, where $A \equiv Lq_{sat}/R_aT$ and $B \equiv (R_a/c_{p,a})(L/R_cT)$. q_{sat} is the saturation mass concentration of the condensable substance, which is approximately the same as the saturation mass mixing ratio in the dilute case. What approximations do you have to make in order to recover this formula from the full nondilute formula given in the text? (*Hint:* Lq/RT can be large even if q itself is small).

Note that both A and B can be written as functions of $\ln T$ and $\ln p$, so if you define y to be the former and x to be the latter, we have a differential equation of the form $dy/dx = F(x, y)$. Write a computer program that solves this differential equation using the Euler and Midpoint methods. (Optionally, you may want to try Runge-Kutta). Show solutions for the case of water vapor in Earth air, with surface temperature set to $250K$, $300K$ and $320K$. Compare with the dry adiabat. Examine the values of q in your solutions and comment on the validity of the dilute approximation. Try this out for some other combination of gases (e.g. CH_4 in N_2 with surface pressures of around $1.5bar$ and temperatures of $80K$ to $100K$, approximating Titan conditions).

You may assume that L is independent of T, with the exception that you should use the latent heat for gas to liquid or gas to solid according to whether the temperature is above or below the freezing point. Do your results change much if you implement this refinement in the case of water vapor?

In this problem you should not use any canned numerical analysis package; you should cook up your own from scratch.

Problem 2.1.53 *Computing the Moist Adiabat*

In this computer problem you will numerically integrate the equation determining the moist adiabat for a mixture of a noncondensable and condensable component, use your routine to reproduce Figure **??** in the text, and then go on to further explorations. Besides basic programming skills, you will need to know how to find numerical solutions to an ODE on the computer.

In Problem 2.1.49 you already wrote a function which computes the slope $d\ln T/d\ln p_a$ in terms of arguments $\ln T$ and $\ln p_a$ and the thermodynamic constants. For generality, you should allow for an arbitrary choice of noncondensable and condensable substances, rather than hard-wiring in the thermodynamic constants. You should assume that L is constant. Test out your slope function in some critical limits to make sure it is behaving properly.

Using the slope function, write a program to determine the moist adiabat for the general non-dilute case, using ODE integration. Recall that, as described in the text, you find the profile $T(p)$ using a two-step process: first you compute $T(p_a)$ by using an ODE integrator, then you use the result to compute the total pressure from the formula $p = p_a + p_{c,sat}(T(p_a))$. Using this procedure, you essentially make a list of values $T(p_a)$ and the corresponding $p(p_a)$, which can then be plotted or analyzed. Verify that this procedure automatically reduces to the one-component moist adiabat for hot conditions where $p_a \ll p_{c,sat}(T)$.

Now, use your system to reproduce the graph of the moist adiabat given in the text (Figure **??**). Extend the results to higher temperature, and compare with what you would have gotten with a pure steam atmosphere.

Python Tips: The integration to find the moist adiabat is implemented in the Chapter Script `MoistAdiabat.py`. It makes use of `gas` objects to allow for generality in the choice of condensable and noncondensable components. The same computation is implemented as a callable object `MoistAdiabatComp` in the `phys` courseware module. The object implementation allows interpolation to a fixed pressure grid, which makes intercomparison of calculations for different parameters easier, and also facilitates comparison with data. You should try to implement the moist adiabat computation yourself before taking a look at the courseware implementations.

Problem 2.1.54 Consider an atmosphere consisting of a saturated mixture of condensable water vapor with noncondensable Earth air. The way we have been setting up the calculation, the partial pressure of noncondensable air, p_a is specified at the ground, and the total pressure p (including water vapor) is computed. From the hydrostatic relation, the total mass of the atmosphere per square meter is p/g, but the mass of air is not p_a/g and the mass of water is not p_w/g because water vapor is not well-mixed, i.e. $p_w/(p_w + p_a)$ is not constant.

Write a function to compute the total mass of water vapor and of air in Earth's atmosphere as a function of p_a at the surface and the surface temperature. Explore the behavior with p_a fixed at $1bar$, as the temperature ranges from $270K$ to $400K$. How much does the mass of noncondensing air vary with surface temperature? Why does it vary? Compare the mass of air with p_a/g, and give an argument explaining the sign of the discrepancy. How does the mass of water in the atmosphere compare with the estimate provided by p_w/g, which is accurate when water vapor dominates the atmosphere?

Hint: If $q(p)$ is the mass concentration of a substance, then the mass of the substance per unit area is $\int_{p_s}^{0} q(p)dp/g$. The integral can be approximated by a sum over layers. In order to compute $q(p)$ from saturation vapor pressure, you will need to make use of the numerical evaluation of the moist adiabat $T(p)$ which you carried out in one or more of the previous problems.

Python Tips: You can do this problem by modifying the chapter script `MoistAdiabat.py`, or by using the `phys.MoistAdiabat` class to make a function to compute the moist adiabat and mass concentration, then summing up the output.

Problem 2.1.55 *Comparison of tropical Earth soundings with the moist adiabat*

Pick a few tropical Earth soundings from the data sets analyzed in Problem 2.1.21. and see how well they match the moist adiabat starting at some pressure p_1 off the ground. You will need to experiment a bit to find the best value of p_1; loosely speaking, it corresponds to a boundary layer height, or in some cases a lifted condensation level. At what pressure level do the sounding start to become warmer than the moist adiabat? This level provides an estimate of the level reached by convection. The convection level, estimated in this way, will always be below the level at which the temperature minimum occurs. Compare the pressures of the two levels in a few soundings selected from the tropical dataset.

Look at the relative humidity data in the soundings. Do the observations follow the moist adiabat even when the atmosphere is substantially undersaturated? The soundings on March 24, 25,26,27, which were taken in the subtropics, provide a particularly clear indication of this behavior.

Finally, examine some midlatitude soundings from the data sets analyzed in Problem 2.1.22. Where is the temperature minimum? How well can you fit the data below the temperature minimum with a moist adiabat?

Problem 2.1.56 *Moist adiabat on Titan*

The atmosphere of Titan consists of a mixture of N_2 and CH_4 (Methane). The methane is condensable at Titan temperatures but for the purposes of this problem the nitrogen can be regarded as non-condensable. (In reality, it does condense a little bit in Titan's upper troposphere). Assume that there is sufficient methane ice or liquid at the surface to maintain methane saturation.

Compute and plot the moist adiabatic $T(p)$ for Titan assuming $1.5 bar$ of N_2 at the ground. You may also assume that methane is saturated throughout the atmosphere (requiring a reservoir of methane at the ground). Plot and discuss results for the mixing ratio as a function of p as well. Discuss results for three different cases: (a) $T = 95K$ at the surface, as for Titan at present, (b) a hot Titan, with $T = 120.K$, and (c) a cold titan, with $T = 80K$ Discuss how the adiabat compares with the dry adiabat and the single-component methane condensing adiabat.

Compare the lower portion of the Huygens Titan sounding to the dry N_2 adiabat and to the saturated moist adiabat based on the observed low level temperature and pressure. Which adiabat provides the better fit to the data? How accurate must the temperature measurement be to distinguish the two? (The data you need is in the `Titan` subdirectory of the Workbook datasets for this chapter.)

Python Tips: You can modify the chapter script `MoistAdiabat.py` for use with a mixture as defined above. Gas data for for N_2 and CH_4 are both in the `phys` module, under the names `N2` and `CH4`. Alternately you can use the `MoistAdiabat` object in the `phys` module.

Problem 2.1.57 Find the moist adiabat for a mixture of $1 bar$ of N_2 (measured by surface pressure) with CO_2 in saturation, in the surface temperature range $200K$ to $300K$. How does the surface pressure depend on temperature?

Problem 2.1.58 As a challenging variant of Problem 2.1.57, consider the case where there are $2bars$ partial pressure of CO_2 at the surface, 1 bar of N_2 and where the surface temperature is warm enough that the CO_2 is unsaturated at low levels. Find the lifted condensation level, and patch the result at this level to the numerically determined moist adiabat further aloft. This is considered by many to be a model of the atmosphere of Early Mars.

Problem 2.1.59 *Moist adiabat on gas giants*

The atmospheres of the gas giants are mostly H_2, mixed with small quantities of various condensable substances. Explore the behavior of the moist adiabat for H_2 mixed with each of the following condensables taken one at a time: H_2O, CH_4, NH_3. For what range of temperatures and pressures does each of these substances appreciably alter the dry H_2 adiabat? For each case, restrict attention to the regime in which the molar concentration of the condensable remains under 10% .

For more accuracy, you should take into account that the noncondensing part of gas giant atmospheres is actually a mixture of H_2 and He, just as present Earth air is mostly a mixture of N_2 and O_2. For Jupiter, He makes up about 10% of the noncondensable total, whereas for Saturn it is about 3%. What do you need to modify in your calculation in order to incorporate the effect of He? Discuss how much difference this makes in the CH_4 case.

Chapter 3

Elementary radiation balance problems

3.1 Workbook

Python tips: The `phys` module contains Planck's constant (`phys.h`), the speed of light (`phys.c`) and the Planck function (invoked as `phys.B(nu,T)`).

3.1.1 Basic concepts

Problem 3.1.1 A comet hits the Earth with a speed of $15km/s$. What mass does the comet have to have in order for the energy delivered by the impact to equal the solar energy received by the Earth in one day? (Assume the albeo is 30%). If the comet has the same density as water, what would its radius be?

Problem 3.1.2 With the present solar luminosity, the Earth absorbs solar energy at a rate of about $240W/m^2$, averaged over the planet's surface. Suppose that the Earth had no way of losing energy. Assuming the absorbed energy to be uniformly distributed over the entire mass of the planet, how much would the Earth heat up after one million years? After one billion years? For the purposes of this problem, you may assume the Earth to have a mean specific heat of 840 J/kgK, which is the value for basalt rock at surface pressure. The mass of the Earth is $6 \cdot 10^{24}kg$.

Problem 3.1.3 Estimate how bright it would look in the daytime on the surface of Neptune's moon Triton. Do this by comparing the flux received from the Sun at Triton with the illumination provided by a candle carried into a dark room. A candle puts out about 1 W of visible light. Assume the candle light bounces off a light-colored wall at a distance of d meters, and compute the flux from the candle which hits the wall. How far would the wall have to be in order to look about as bright as the surface of Triton?

Problem 3.1.4 The V-filter used by astronomers corresponds to the peak sensitivity of human vision. It has a central wavelength of 0.55 μm and a bandwidth of about 0.1 μm. If the flux density in wavenumber space passing through this band is measured in $W/(m^2 cm^{-1})$ then the constant C used in defining zero magnitude for this band is -29.836 . (See Problem 1.1.14 for the definition of magnitude). Note than when defining filtered magnitudes, flux density is used in the formula rather than flux itself. This makes the definition less sensitive to the filter bandwidth.

A telescope with a collector area of 1 m^2 observes a star with apparent V-magnitude M_V. Make a table showing how many V-band photons strike the collector per second, for $M_V = 0, ..., 20$.

Hint: Make sure to convert to frequency in Hz when using Planck's constant, and to convert the bandwidth to units compatible with the flux density. This problem involves a mishmash of different units, but that's just like real life; at least you have been spared learning what a *Jansky* is.

Problem 3.1.5 In the text we derived the $1/r^2$ law for solar radiation by dividing the net power output of a star by the area of a sphere of radius r with the star at its center. In this problem, you will rederive the same result by summing up the blackbody radiation coming from each patch of the star's visible surface, making use of the angular distribution of blackbody radiation. This problem tests your understanding of the Planck density and the way the angular distribution is represented. It also provides a way to compute the solar constant in cases where the brightness is not uniform across the star's surface.

Consider an observer at a distance $r \gg r_{\circledast}$. You need to compute the power incident on a small flat patch P with area dA, oriented perpendicular to the line joining the observer to the center of the star. Each patch $P_\circledast$ of the visible surface of the star, having area $dA_\circledast$, contributes a flux (per unit frequency) $B(\nu, T_\circledast) dA_\circledast d\Omega \cos\theta$ to the net flux going through P. In this formula, $d\Omega$ is the (small) solid angle made up by the set of rays from $P_\circledast$ which go through P, and θ is the angle between the normal to the star's surface at $P_\circledast$ and the line joining the center of $P_\circledast$ to the center of P. Complete the derivation by responding to the following:

- Why do we need the factor $\cos\theta$?
- Find an expression for $d\Omega$ in terms of dA and r.
- Find an expression for $\cos\theta$ in terms of the position on the star's surface.
- Integrate $B d\Omega \cos\theta$ with respect to $dA_\circledast$ over the visible surface of the star. Note that since $r \gg r_\circledast$, the visible surface (i.e. the part of the surface from which some rays pass through the patch P) is approximately a hemisphere.

If you did everything properly, you should obtain the same answer as we got using the previous method, namely $\pi B r_\circledast^2 / r^2$.

3.1.2 The Planck Function

Problem 3.1.6 Consider a patch of the photosphere of the Sun with an area of 10 m^2. The patch is small enough that it can be considered flat. Assume that the temperature of the photosphere is $6000K$ and that it radiates like an ideal blackbody. Now consider all the light with wavenumbers between 10000 and 10100 cm^{-1}, which leaves the patch in directions making angles between 45 degrees and 50 degrees with the normal to the patch. How much power (in W) is contained in the light meeting these requirements? What is the rate (in W) at which energy is carried *through* the patch by such light? For the purposes of this problem, you may assume the Planck function to be constant within the specified wavenumber band.

Problem 3.1.7 What is the total power radiated by a blackbody sphere of radius $1m$ having uniform temperature $300K$, in the wavenumber range between $500cm^{-1}$ and $750cm^{-1}$. You may assume the Planck density to be approximately constant over this range. Compare this to the total power radiated over all wavenumbers. *Note:* Blackbody radiation is emitted with equal intensity in all directions, so you must remember to take this into account in computing the total energy flux coming from the surface.

Problem 3.1.8 Compute the *wavenumber* of maximum emission of an object with temperature $200K$. What is the rate of energy emission from each square meter of the object's surface, in the wavenumber band extending $50cm^{-1}$ on either side of the maximum? You may assume for the purposes of this calculation that the Planck function is constant over this range of wavenumbers. How good is this assumption?

Problem 3.1.9 Write down the Planck density corresponding to the case in which position in the electromagnetic spectrum is measured by $\ln\lambda$, where λ is the wavelength. Estimate the value of $u = h\nu/(kT)$ for which this density has its peak. (You will probably need to do a small numerical calcuation to do this). How does your result change if you use the log of the wavenumber or frequency as the coordinate instead?

Problem 3.1.10 *Color index and stellar temperature*

Suppose we observe the flux density from a star at two different frequencies ν_1 and ν_2, with $\nu_2 > \nu_1$. Let the measured flux densities be F_1 and F_2. Suppose further that the star radiates like an ideal blackbody with temperature $T_\circledast$. The two flux densities will then be proportional to the Planck function at the corresponding frequencies, multiplied by the inverse-square attenuation of the starlight. Write the flux densities as astronomical magnitudes using $M_j = C_j - 2.5 \log_{10} F_j$, where C_j the constant defining zero magnitude for each band. Show that $M_2 - M_1$ is independent of the distance from the star. This difference is called a *color index.* Plot the color index as a function of $T_\circledast$ for ν_2 corresponding to the astronomer's blue ("B") band at $0.44\mu m$ and ν_1 corresponding to the visual ("V") band at $0.55\mu m$, and show that the magnitude difference is a monotonically decreasing function of temperature. Find analytic expressions in the low-frequency and high frequency limit, and use these to get an approximate expression for $T_\circledast$ as a function of the color index. Given an observation of the color index alone, how would you know which, if any, of these two approximations is applicable? *Hint:* In the low frequency case you will need to retain one term beyond the lowest order expansion in frequency in order to get a dependence of the color index on $T_\circledast$.

Astronomers define color indices for specific standard filter bands. The B-V color index is very commonly reported as a proxy for the temperature of an object. Various empirical fits are available, which take into account the actual standard filter characteristics. For example, for temperatures between $3000K$ and $9000K$, the empirical relation is $M_B - M_V = -3.684 \log_{10} T + 14.551$. Compare this fit to the results you got in your calculation above, which in essence assume infinitely narrow filters. In making this comparison, choose $C_1 - C_2$ so that the agreement is exact at $6000K$.

Problem 3.1.11 *Bolometric Correction*

Problem 1.1.14 introduced the concept of bolometric magnitude. Earth based astronomers rarely measure the bolometric magnitude directly, because the atmosphere absorbs much of the spectrum and it is hard to take accurate all-wave flux measurements. Direct measurement of bolometric magnitude is becoming more common with the advent of space-based telescopes, but it is still common for astronomers to report V-magnitudes and add a temperature-dependent correction to convert to bolometric magnitudes assuming a blackbody spectrum. This term is called the *bolometric correction*, denoted by BC

Suppose that the observed flux density near frequency ν_o is F_o. Compute the total flux F_{tot} integrated over all wavenumbers, assuming that the spectrum is proportional to a Planck spectrum with temperature $T_\circledast$. Show that for any given ν_o, F_{tot}/F_o depends only on $T_\circledast$. Using the definition of magnitude, this ratio is converted to a bolometric correction using $BC = C_{bol} - C_o - 2.5 \log_{10}(F_{tot}/F_o)$, where C_{bol} and C_o are the constants used to define the zero-magnitude point for the bolometric and filtered fluxes, respectively. For V-magnitudes this difference is about 10.86. Plot the bolometric correction as a function of $T_\circledast$, for ν_o corresponding to the V-band at 0.55 μm.

The following empirical fit gives the bolometric correction taking into account the actual filter characteristics of the standard V-filter:

$$BC = -8.499x^4 + 13.421x^3 - 8.131x^2 - 3.901x - 0.438, x \equiv \log_{10} T_\circledast - 4 \tag{3.1}$$

Compare this with the narrow-filter calculation you did above.

Problem 3.1.12 Show that if both ν_1 and ν_2 are in the frequency range where $h\nu/kT \ll 1$, then the total flux per steradian emitted between ν_1 and ν_2 is $\frac{2}{3}(kT/c^2)(\nu_2^3 - \nu_1^3)$.

Problem 3.1.13 *Microwave brightness of Venus*

The thick CO_2 atmosphere of Venus is nearly opaque in the infrared, so observations of Venus in the infrared spectrum provide no direct information about the temperature of the ground. However, the atmosphere is nearly transparent to microwave radiation, so the microwave emission of the planet can be used to infer its surface temperature. Indeed, this is how it was inferred in the 1960's that the surface of the planet was hotter than a pizza oven, rather than the steamy jungle world earlier science fiction authors had envisioned.

Assuming the ground to radiate like a blackbody in the microwave, compute the net power (in W) radiated by Venus in the wavelength band from 1 millimeter to 100 millimeters, if the surface temperature is uniform at $737K$. What would the radiated power be if the surface temperature were $300K$ instead? *Hint:* You can use the result from Problem 3.1.12. Why is this valid?

At its closest approach, Venus is about 41 million km distant from Earth. Using the $1/r^2$ law, what microwave energy flux (in W/m^2) would be seen from Earth orbit by a microwave antenna directed toward Venus? How much microwave power would be collected by an antenna with area $100m^2$?

Problem 3.1.14 Compute the total power radiated by a person with a normal body temperature of $37C$. Why is this so much greater than the typical daily energy consumed by a person in the form of food (equivalent to about $100W$)? Next, compute the power radiated by the person in the visible wavelength band (about .5 to 1 microns). Approximately how many visible light photons per second are radiated? About how long would you have to wait before the person emitted a *single* ultraviolet photon (about .1 micron wavelength)?

For the purposes of estimating the surface area needed in this problem, you may assume that the person is shaped approximately like a rectangular prism, with height $1.5m$, width $.5m$ and depth $.25m$.

Problem 3.1.15 Derive an approximate form of the cumulative blackbody emission spectrum, valid for large values of $h\nu/(kT)$. *Hint:* Write the cumulative spectrum as $\sigma T^4 - I_1(h\nu/(kT))$, and use the fact that the dominant contribution to the integral I_1 comes from the vicinity of the lower limit of integration, $h\nu/(kT)$.

Problem 3.1.16 Estimate the total power radiated by the Earth at wavelengths equal to or shorter than visible light (about 5 microns. For the purposes of this problem, assume that the Earth has a uniform radiating temperature of $255K$.

Problem 3.1.17 In the text we needed to evaluate the integral $\int_0^\infty (u^3/(e^u - 1))du$ in order to determine the Stefan-Boltzmann constant. Using a Taylor series expansion of the integrand, show that the integral is equal to $6\zeta(4)$, where

$$\zeta(n) = \sum_{j=1}^{\infty} \frac{1}{j^n} \tag{3.2}$$

Hint: Multiply the numerator and denominator of the integrand by e^{-u} and then Taylor expand in e^{-u}. Equivalently, the expansion can be done as the geometric series $1 + a + a^2 + ...$ where $a = e^{-u}$. Note that the series is convergent for all $u > 0$ because $e^{-u} < 1$

3.1.3 Basic Energy Balance

Problem 3.1.18 You are designing a spherical planet to be placed at the orbit of Mercury. The planet will have a nitrogen atmosphere that has no effect on the infrared radiated by the planet, but which is dense enough and mixes heat rapidly enough that the entire planetary surface is isothermal. What albedo should the planet have in order for its surface temperature to be a comfortable $300K$?

Problem 3.1.19 A cylindrical space station with length h and radius r is in an orbit about the Sun at a distance where the solar constant is $L_\circledast$. The space station has zero albedo in the visible range and radiates as a perfect blackbody. The flow of air inside keeps the entire station at the same temperature, and the skin is a good conductor of heat, so that its temperature is the same as that of the interior. The orientation of the station is such that the axis of the cylinder is always perpendicular to the line joining the center of the station to the center of the Sun. Find an expression for the temperature of the station. Put in numbers corresponding to the mean Solar constant at Earth's orbit, assuming $r = h$.

Now suppose that the equipment in the interior of the space station consumes 1 megawatt of solar-generated electrical power, which is dissipated as heat. How much warmer would this make the station once the equipment was turned on? To get rid of this excessive heat, you are to design a radiator, which is a large, thin flat plate heated by pumped water from the space station so that its temperature is the same as the interior of the space station. The radiator is perfectly reflective in the visible range, but acts as a perfect blackbody in the infrared range. How large should the radiator plate be in order to get rid of the excess heat? For the purposes of this part of the problem you may assume $r = h = 50m$.

Problem 3.1.20 Compute the equilibrium temperature at the subsolar point of Europa, which is in orbit around Jupiter and therefore has the same solar constant as that planet. The greenhouse effect of Europa's tenuous atmosphere can be neglected. For the purposes of this problem, you may assume that the albedo of Europa is .67 . Assuming Europa to have a water-ice surface, what would be the saturation vapor pressure of the water vapor atmosphere immediately above the subsolar point? Suppose that there is some methane, carbon dioxide and ammonia mixed in with the ice. What would be the partial pressures of these gases?

Problem 3.1.21 Consider a planet covered in water ice with a uniform albedo of .7 . The planet is tide-locked, so that the same face always points to the Sun; the other side is in perpetual night. The atmosphere has negligable greenhouse effect. Compute the solar constant needed to begin melting the ice under the following three alternate scenarios: (a) The atmosphere is so efficient at transporting heat that the entire surface of the planet (dayside and nightside) has the same temperature, (b) The atmosphere is only moderately efficient, so that the dayside temperature is uniform but essentially no energy is carried away to the nightside, (c) There is no atmosphere, so that each bit of the planet's surface is in equilibrium with the solar radiation it absorbs.

Problem 3.1.22 The physicist Freeman Dyson has speculated that an advanced civilizations might enclose its star within a spherical shell of material,and live on the inner surface. This gives them more living space, and avoids wasting all that starlight that would ordinarily escape to space. Consider a red dwarf star with photospheric temperature $4000K$ and radius $200{,}000km$. What should the radius of the Dyson sphere be in order to have an inner-surface temperature of $300K$? Assume that the sphere material is a perfect blackbody with zero albedo at all wavelengths, and that it is a good conductor of heat so that the inner surface and outer surface temperature are identical. You may also assume that the temperature of the photosphere remains unchanged after the Dyson sphere is built. If you were an astronomer looking for such a Dyson sphere, in what part of the electromagnetic spectrum should you look? What radius should the Dyson sphere have if it is instead made of a good thermal insulator, so that the outer surface is much, much colder than the inner surface?

Challenge question: In either case, is it actually reasonable to assume the photosphere temperature stays unchanged? If you think it should change, what value should it take on assuming: (a) the fusion energy output of the star remains unchanged, and (b) the radius of the Dyson sphere is adjusted so that the inner temperature is $300K$ as before.

Problem 3.1.23 The discovery of a planetary system orbiting the red-dwarf star Gliese 581 was announced in 2007. This system is of particular interest, because two of the planets (581c and 581d) have masses just a few times that of Earth, and so are presumably solid (ice or rocky) bodies rather than gas giants. The star has a luminosity .013 times that of the Earth's Sun. Gliese 581c orbits at a mere .073 A.U. (Astronomical Units) from its star, and 581d orbits at .25 A.U. The temperature of the photosphere of Gliese 581 is $3480K$.

Compute the equilibrium temperatures of these two planets, assuming that they are isothermal spherical bodies with an albedo of .3 , and assuming that the atmosphere has no greenhouse effect. What is the gap between the median-emission wavenumber of the star and that of each of the planets? Compare this to the situation of Mercury in our own solar system. Based on the data given, estimate the radius of the star in the Gliese system.

Problem 3.1.24 *Freeze-out time of magma ocean*

This problem is an idealization of the calculation of the freeze-out time of a magma ocean such as might have occurred on Earth following the moon-forming impact. Earthlike magma freezes at $2036K$ at near-surface pressures. We'll assume the entire planet is liquid and convecting vigorously, and that the entire planet is isothermal at this temperature until enough energy has been removed to freeze the whole planet solid. The latent heat required to freeze the magma is on the order of $3 \cdot 10^5 J/kg$. Under these assumptions, compute the time needed for the magma ocean to freeze, assuming the radius of the planet to be a, the density to be ρ and the atmosphere to be transparent to infrared. The latter assumption implies that the planet radiates at a rate of σT^4 per unit surface area, where T is the surface temperature (held constant at the melting point). Estimate the time, in years, for this to happen on the Earth, assuming a density of $5500 kg/m^3$, and verify the statement in Chapter **??** that this is a matter of a few hundred years. In doing this calculation, does it matter much how much solar radiation the Earth absorbs? How does the answer change if your planet is bigger?

Note that this calculation produces an underestimate of the time for the entire planet to freeze out, and an overestimate of the time required for a crust to first form. The reason is that in reality heat will be removed first from an upper layer of the magma ocean, allowing a crust to form at some point less than the freeze-out time for an isothermal magma ocean. Once the crust forms, the insulating properties of solid rock will greatly reduce the heat flow from the interior, extending the time needed for the rest of the planet to solidify.

Problem 3.1.25 *Energy balance of satellites*

A spherical satellite with radius a_1 is in a circular orbit at a distance r_1 from its planet, which is also spherical. The planet has radius a_o and is located a distance r_o from its star;$r_1 \gg a_o$ and $a_o \gg a_1$. Both the planet and the satellite are perfectly spherical,both are isothermal. The planet has zero albedo in the shortwave (solar) spectrum, but the satellite haz a shortwave albedo of unity so that it is not directly heated by the star. The satellite absorbs perfectly in the infrared spectrum.

Derive an expression for the temperature of the satellite, and compare this to the temperature the satellite would have if it had zero shortwave albedo and were heated directly by the sun. Based on these results, do you think one needs to include the infrared emission of Saturn when computing the temperature of Titan (which orbits at a distance of about one million km from Saturn)?

How do your results change if you relax the assumption $r_1 \gg a_o$, i.e. you allow the orbit to be so close that the planet "fills the sky"? (This regime is unlikely for natural satellites, but is quite common for artificial satellites in low orbit about the Earth or other planets).

Problem 3.1.26 A sphere rests on an infinite horizontal plain with uniform temperature T. Both the sphere and the plane are perfect blackbodies. The sphere does not absorb any sunlight. There is no atmosphere, so the only tranfer of heat is by radiation. What is the temperature distribution of the surface of the sphere if it is a perfect insulator so no heat is transferred from one place to another within the sphere? What is the temperature of the sphere if it is filled with a well-stirred fluid which keeps the surface isothermal?

3.1.4 The Greenhouse Effect

Problem 3.1.27 For Jupiter, the observed OLR is $14.3W/m^2$. Compute the effective radiating temperature. Referring to the Jupiter temperature profile given in Chapter **??**, estimate the effective radiating pressure of Jupiter. Is there more than one possible value? Which of these do you consider more likely?

Problem 3.1.28 *Early Mars Climate*

The surface of Mars has river-like channel networks that suggest that at sometime in the past Mars was warm and wet. Some of these features date back to a time several billion years ago when the Solar luminosity (and hence the Solar constant at the orbit of Mars) was about 25% less than its current value. One school of thought states that the mean surface temperature of Mars would have to have reached values at the freezing point ($273K$) or above to account for such features.

Assume that Early Mars had a dense atmosphere that kept its surface temperature approximately uniform, and that the planet had an albedo similar to the present one (about .25). Compute the greenhouse effect – as measured by p_{rad}/p_s – needed to bring the surface temperature of Early Mars up to the freezing point under the following three alternate assumptions regarding the vertical profile of temperature:

- (a)A pure CO_2 atmosphere on the dry adiabat.
- (b) A mixture of equal parts (molar), of CO_2 and N_2 on the dry adiabat

Note that if κ is the grey absorption coefficient for the gas, then $p_{rad}/p_s = g/\kappa p_s$, so you are in essence computing the surface pressure of CO_2 needed to make Mars warm and wet. In Chapter **??** this problem will be done with real-gas radiation physics, and it will seen in that case that the effects of CO_2 condensation on the profile can significantly inhibit warming. Why might CO_2 condensation aloft affect the warming?

For the purposes of this problem, you may assume that the ground temperature is equal to the low-level air temperature.

Problem 3.1.29 Consider an atmosphere whose temperature profile is on the dry adiabat with $R/c_p = \frac{2}{7}$, all the way down to zero pressure (infinite altitude). The surface temperature is $300K$ and the surface pressure is $10^5 Pa$. The gravity is the same as Earth's gravity, $9.8m/s^2$. We mix in a hypothetical greenhouse gas with mass concentration q. The greenhouse gas has the property that its absorption coefficient κ is $1m^2/kg$ for wavelengths between 15 *microns* and 18 *microns*, and is zero elsewhere. Mixing in the greenhouse gas does not change the temperature profile, and you may assume that q is small enough that the change in surface pressure may be neglected. The radiating pressure for any given κ is estimated using $\kappa q p_{rad}/g = 1$, and the radiating temperature is the temperature at this pressure level. Using this estimate, answer the following questions.

The radiating temperature, or *brightness temperature*, for any give frequency ν is the temperature T_{rad} such the actual emission to space is equal to the blackbody emission $\pi B(\nu, T_{rad})$. Sketch a graph of the brightness temperature vs frequency, showing curves for a range of q from zero to 0.1. Compute a graph of the way the spectrum of outgoing radiation (as observed from space) depends on frequency, for the same range of q. Finally, graph the behavior of the *net* outgoing radiation (i.e. the integral of the curves in the previous question) as a function of q.

Hint: You can answer the last question using the fact that the integral of $\pi B(\nu, T)$ over all ν is σT^4, together with a numerical evaluation of the integral over the 15 to 18 *micron* band. When the frequency interval is small enough that B is essentially constant over the frequency range, then the numerical integral should be approximately $\pi B(\nu_o, T)\Delta\nu$, where ν_o is the frequency at the center of the band. How well does this approximation match the numerically computed value?

Problem 3.1.30 Consider an atmosphere in which, for some reason, the temperature *increases* with height, according to the formula $T(p) = T_s + a \cdot (p_s - p)$ with a a positive constant. Suppose that the addition of a greenhouse gas elevates the radiating level to $p_{rad} < p_s$. Find an expression for the dependence of the surface temperature on p_s, assuming that the atmosphere is in equilibrium with an absorbed solar flux $S \cdot (1 - \alpha)$ per unit surface area of the planet.

Problem 3.1.31 Taking into account its observed albedo, Titan absorbs $2.94W/m^2$ of solar radiation (averaged over its entire surface). The observed surface temperature is 95K. If you assume that the temperature profile of the atmosphere is given by the dry adiabat for pure N_2 having a surface pressure of $1.5bar$, what would the radiating pressure for Titan have to be in order to account for the observed surface temperature?

Problem 3.1.32 *Analysis of the Earth's observed radiation budget*

To do this DataLab, you will need to know how to read tabular data from a text file using whatever software you are using for data analysis, and to perform simple arithmetic manipulations of the data. This DataLab can be done using most common spreadsheets and many kind of plotting software, as well as in programming languages such as `Python` or `Matlab`.

The course data directory contains a number of plain text files containing data for the zonal mean (i.e. mean along latitude circles) of quantities measured by the Earth Radiation Budget Experiment, and also surface temperature data. These are contained in the subdirectory `Chapter3Data/ERBE/`. Carry out the following analyses on the data:

For July 1988, carry out the following analyses:

- From the *OLR* data, compute the effective radiating temperature T_{rad} as a function of latitude

- Plot it on the same graph as the actual surface temperature. Also, make a plot of the difference between T_{rad} and T_s. Where is the difference large? Where is it small? Discuss your results, taking note of where it is summer and where it is winter.

- Estimate the radiating pressure p_rad assuming the surface pressure to be $1000mb$ and assuming the atmosphere to be on the dry adiabat. Plot p_{rad} as a function of latitude. Note that this will be an overestimate of p_{rad} because the real atmosphere is closer to the moist adiabat.

Python Tips: The chapter script `ERBEplot.py` gives an example of how to read the data, plot it, and perform simple manipulations of the data.

Problem 3.1.33 *The atmosphere as heat engine*

A *heat engine* is a device that makes use of the temperature difference between two reservoirs in order to do mechanical work. It transfers energy from the hot reservoir to the cold reservoir; the difference between the energy drawn from the hot reservoir and the energy deposited in the cold reservoir is the mechanical work that can be done by the device. For a cycle that starts and ends at the same (p, T) state, Eq. **??** implies that $W \equiv -\oint \rho^{-1} dp = \oint T ds$. The final step proceeds from the fact that $\delta Q = T ds$, where s is the entropy, and the fact that $\oint c_p dT = 0$ around a closed loop if c_p is constant. (If c_p is not constant the same result applies, but must be re-phrased in terms of internal energy, defined by $dU = c_p dT$.)

A planetary atmosphere works like a heat engine working between the temperature of the star's photosphere ($6000K$ for our Sun) and the temperature of space (essentially zero). Only a small part of this temperature difference can be tapped to do mechanical work, however. Consider a dry atmosphere consisting of an ideal gas which is transparent to the incoming solar radiation. All the incoming energy is absorbed at the planet's surface, whereafter it is transferred by heat conduction to the air in contact with the ground. Using the above thermodynamic relation, show that if the atmosphere is transparent to infrared as well, so that the only energy loss to space is by radiation from the ground, then the system cannot do any mechanical work. *Hint*: Think about the surface heating up by a certain amount in time Δt owing to solar absorption, and then cooling down by the same amount owing to radiation of infrared.

Now suppose that the atmosphere is not transparent to infrared, and that the radiation escapes to space from a pressure $p_{rad} < p_s$, where p_s is the surface pressure. Consider a cycle where the atmosphere is maintained on a dry adiabat with surface air temperature equal to ground temperature by the following process. Solar absorption increases the ground temperature and low level air temperature by an amount ΔT in a time ΔT. The air, whose temperature is now $T_s + \Delta T$ becomes buoyant and rises *adiabatically* (i.e. at constant s) to the pressure p_{rad}, where it cools down by radiation until its temperature is equal to the ambient adiabat $T_s \cdot (p_rad/p_s)^(R/c_p)$ After that it sinks adiabatically back to the ground, where the process starts all over again. Both the heating and cooling occur at constant pressure. Sketch this cycle on thermodynamic diagrams in $(p, 1/\rho)$ and (T, s) space.

Compute the work done in the course of the preceding cycle, and describe how it depends on p_{rad}/p_s. This is the work available to drive the atmospheric circulation. The solar energy per unit mass put into the parcel in time Δt is $c_p \delta T$. Compute the efficiency of the atmospheric heat engine by dividing W (the work done per unit mass) by this quantity. Show that the thermodynamic efficiency increases as the greenhouse effect is made stronger.

3.1.5 Ice-albedo feedback, hysteresis and bifurcation

Problem 3.1.34 *Computing the Ice-Albedo Hysteresis Diagram*

Calculations with a complete real-gas radiation simulation indicate that, for a CO_2 concentration of $300 ppmv$, and with an atmosphere on the moist adiabat, a reasonable fit to the actual OLR curve in the range of $220K$ to $310K$ is the linear fit $OLR(T) = a+b\cdot(T-220)$ where $a = 113W/m^2$ and $b = 2.177W/m^2K$. Compute the ice-albedo hysteresis diagram giving the set of equilibrium temperatures as a function of the solar constant $L_\circledast$. Note that there is a simple trick for getting the bifurcation plot. The equation determining the equilibrium is $\frac{1}{4}L_\circledast(1-\alpha(T)) = OLR(T)$ Instead of specifying $L_\circledast$ and finding the T that satisfy the equation, we can re-write the equation as

$$L_\circledast = 4\frac{OLR(T)}{1-\alpha(T)} \tag{3.3}$$

Now, if we call the right hand side $G(T)$, then $G(T)$ gives the unique value of $L_\circledast$ which supports the temperature T. Hence, to get the bifurcation diagram, you can just plot $G(T)$ and then turn it sideways.

Use the same albedo-temperature function defined in Chapter 3. Assume that the albedo for an ice-free Earth is .2 and for an ice-covered Earth is .6.

Based on your calculations, if CO_2 were held constant how much would $L_\circledast$ have to be reduced from its modern value before Earth was forced to fall into an inevitable snowball state? Using the inverse-square law and assuming a circular orbit, compute how far out from the Sun the Earth would have to be displaced (relative to its present orbit) to achieve this solar constant. Conversely, how close to the Sun would you have to place the Earth before a Snowball state became impossible? *Note:* The assumption of fixed CO_2 is very unrealistic, since tectonically active planets with water have a way of adjusting CO_2 in response to changes in the solar constant. This will be discussed in Chapter **??**.

Python Tips: The script `IceAlbedoZeroD.py`, found in the Chapter Scripts collection for this chapter, calculates and plots hysteresis diagrams using this method, and also makes various other plots useful in understanding bifurcations due to ice-albedo feedback. You can easily modify this script to solve this and similar problems, by redefining the `OLR(T)` function, and (in other cases) also the albedo function.

Problem 3.1.35 *Daisyworld* This is a variant on a theme due to Lovelock. Daisyworld is a black planet (zero albedo) upon which white daisies (albedo α_d) can grow. The daisies are temperature sensitive – if the planet is too cold, the daisies die out and decompose, leaving the albedo unaltered. The same happens if the planet is too hot. In between, the daisy coverage varies such that $\alpha(T) = \alpha_d \cdot 4(T-T_1)(T_2-T)/(T_1-T_2)^2$, where T_1 and T_2 are the temperatures deterimining the survival range for the daisies.

Assuming $OLR(T) = \sigma T^4$, discuss the possible equilibrium states of the climate and the way they vary as the solar constant is changed. Which solutions are stable? Which are unstable? Does this system exhibit hysteresis as the solar constant increases and decreases?

For simplicity you may assume $T_1 = 200K$ and $T_2 = 300K$ if you want, but you should discuss how the qualitative behavior of the system changes as you vary α_d between zero and unity.

3.1.6 Kirchoff's Law, Emissivity,absorptivity

Problem 3.1.36 Consider a sandwich made of three slabs of material The two outer slabs are blackbodies, with temperatures T_1 and T_2 respectively. Between these two is a slab which is a grey body having emissivity e. What is the temperature of the grey-body slab?

Problem 3.1.37 Suppose that the ground has temperature T_s, which is a given. The ground has unit infrared emissivity. There is one isothermal layer above the ground with emissivity e_1, and another one above that with emissivity e_2. Neither layer absorbs any Solar radiation, and neither layer is affected by any heat transfer mechanism other than radiation.

Find an expression for the temperatures of the two layers. Do not assume that the emissivities are small.

Problem 3.1.38 *Solar-absorbing stratospheric cloud*

Consider an atmosphere which is transparent to solar and infrared radiation, except for a high stratospheric cloud which has emissivity e in the infrared and absorptivity a in the solar spectrum. The cloud is above the region reached by convection, so it is in equilibrium with the absorbed energy from upwelling infrared coming from the planet's surface, the solar radiation which it absorbs directly, and the infrared cooling due to the cloud's emission; it receives no heat at all from fluid dynamical transports. The surface of the planet has constant albedo α, and has unit emissivity in the infrared. The whole system is in equilibrium with an incident solar flux S per unit area of the planet's surface.

Compute the surface temperature and the temperature of the cloud. Under what circumstances does the cloud increase the surface temperature? Under what circumstances does it cool the surface? Note: this "cloud" is somewhat like the Earth's ozone layer, which absorbs ultraviolet from the Sun, but also has significant infrared emissivity.

Hint: First do the problem with $\alpha = 0$ to get a feel for what is going on. Then add in the terms in the budget that come from reflection of solar radiation off the ground.

Problem 3.1.39 Consider an optically thin atmosphere which has a nonzero emissivity e within a very narrow frequency band of width $\Delta\nu$ centered on frequency ν_o. This atmosphere is above a surface having a known temperature T_g, which radiates like a blackbody. What is the temperature of the atmosphere if no convection is allowed to take place? Explore how the temperature varies as a function of ν_o. Does the result depend on $\Delta\nu$? On the emissivity? Put in numbers corresponding to Martian afternoon conditions: $T_g = 230K$, and ν_o corresponding to a wavenumber of 650 cm^{-1} (approximately the center of the principle absorption band of CO_2).

3.1.7 Second-law challenges

These rather open-ended questions provide some food for thought.

Problem 3.1.40 *Do burning glasses violate the Second Law?*

With a magnifying glass on a clear, sunny day, you can focus sunlight to a small dot and achieve extremely high temperatures. The same could be done with a parabolic mirror. Suppose you are conducting this experiment in the vacuum of space, so you needn't need to be concerned with losses due to atmospheric absorption or reflection. Based on the Second Law, what would you expect the maximum achievable temperature to be at the focus? Do magnifying glasses and mirrors indeed conform to this limit? What processes, if any, keep these devices from violating the limit? *Hint: Think about the implications of the Sun's finite angular size, as seen from any given orbit.*

Problem 3.1.41 *Do solar-electric furnaces violate the Second Law?*

One could set up a large array of solar cells in the desert, and then use the electricity generated to power an electric furnace in which the electric energy is dissipated in a small enough volume to achieve an arbitrarily high temperature. Do such devices violate the Second Law?

Problem 3.1.42 *Frequency Doublers*

Green laser pointers make use of a nonlinear frequency-doubling device, which takes near-infrared laser input and doubles the frequency to produce green light. This is not a filter, but a transformer of frequency. If such a frequency doubler could be made lossless so that it didn't consume any energy, could you use it to make a device that violated the Second Law (i.e. an energetically closed device that would spontaneously generate temperature gradients from an isothermal state)? *Hint:* Think about what you could do if such a frequency doubler were used in place of the selective mirrors in Kirchoff's argument for Kirchoff's Law.

Chapter 4

Continuous atmosphere radiation problems

4.1 Workbook

4.1.1 Basic Grey-Gas calculations

Problem 4.1.1 Suppose that the temperature profile in an atmosphere is $T(p) = T_s \cdot (p/p_s)^(1/4)$. (This is a rather contrived temperature profile, but it is chosen to make the algebra simple). The ground temperature is equal to T_s. Compute the OLR and the downward radiation into the ground, assuming the atmosphere to be a grey gas with constant absorption coefficient κ. Discuss the behavior in the limits of large and small κ.

Problem 4.1.2 *Grey Gas OLR computations*

In this computational problem you will explore the grey-gas OLR for the temperature profile

$$T(p) = \max[T_{strat}, T_{sa} \cdot (\frac{p}{p_s})^{R/c_p}] \tag{4.1}$$

where T_{strat} and T_{sa} are constants. This profile follows the dry adiabat until the temperature falls to T_{strat}. At lower pressures (higher altitudes), the temperature is held constant at T_{strat}. This is a primitive model of a stratosphere, in that the temperature is prevented from falling to zero with height. You may assume that the specific absorption cross section of the atmosphere (κ) is independent of pressure and temperature, as well as frequency. Note that this absorptivity enters the problem only through the net optical thickness of the atmosphere τ_∞. If we set $T_{strat} = 0$ the problem reduces to the all-troposphere problem discussed in the text, in which T approaches zero as p approaches zero. The ground temperature T_g may differ from the surface air temperature T_{sa}.

Write a program defining a function that computes OLR in the above situation by evaluating the integral form of the solution to the Schwartzschild equation using numerical quadrature. The function should return the OLR as a function of the arguments T_g, T_{sa}, T_{strat} and τ_∞.

Next, use your OLR function to estimate the temperature of a planet in equilibrium with an absorbed solar radiation of $250 W/m^2$. Do this for $T_{strat} = 200K$ and for optical depth $\tau_\infty = .1, .5, 1.$ and 5.. You may assume $T_g = T_{sa}$

Now explore the behavior of OLR as the optical thickness `tauInf` is varied between small values (say, .1) and large values (say, 100), with $T_{strat} = 200K$. Can you find a simple explanation for the limiting behavior, based on what you know about blackbodies? Now reduce T_{strat} to $100K$ and try again. How does this change the behavior of OLR in the optically thick limit? Why? Next, set T_{strat} to zero. Compare the behavior of the OLR at large `tauInf` with the all-troposphere formula derived in the text. (If you stick with the value of R/c_p for air you will find it helpful to know that $\Gamma(1 + 8/7) \approx 1.07$. What if $4R/c_p$ is an integer?). Recall that the all-troposphere expression was derived assuming that τ_∞ is large. Using your numerical computation of OLR, discuss how the approximation breaks down as τ_∞ becomes smaller. How good is the approximate form for $\tau_\infty = 1$? What is the true behavior of OLR as τ_∞ is made small? Give an explanation for this behavior. Finally, set $T_{strat} = 100K$ again, and compare the OLR with the all-troposphere formula. Where does the computed OLR diverge from the all-troposphere formula? Why? Under what circumstances does the choice of T_g substantially affect the OLR, and what is the nature of the effect?

Note that this problem is somewhat unrealistic, since we keep the stratospheric temperature fixed as optical thickness is changed. In reality, changing the optical thickness would affect the stratospheric temperature and the tropopause. Nonetheless, the problem provides some insight as to how the stratosphere affects OLR.

Python Tips: You can do this problem using a modification of the script `GreyGasFlux.py`

Problem 4.1.3 *Computation of grey gas heating rates*

In Fig. **??** we used a computation of net radiative flux to determine which portions of an all-troposphere atmosphere were subject to radiative cooling, and which would warm up and form a stratosphere. Write a program to compute and plot the vertical profile of net upward flux for a grey gas with temperature profile given by the dry air adiabat throughout the depth of the atmosphere. The radiative fluxes can be most easily computed by evaluating the integral form of the Schwartzschild equation solution using numerical quadrature. Compare the results to the optically thick approximation. Check your routine by reproducing the results in Fig. **??**, which was done assuming a pressure-independent absorption coefficient.

Now re-do the calculation allowing for pressure broadening, i.e. a linear increase of absorption with pressure. The optical thickness in this case is given by

$$\tau(p) = \frac{1}{2}\kappa(p_s)\frac{\cos\bar{\theta}}{g}(p_s^2 - p^2) \tag{4.2}$$

Plot the results and compare with the optically thick limit (modified to allow for pressure broadening). What happens to the stratosphere as the atmosphere is made more optically thick?

Python Tips: Modify `GreyGasFlux.py`.

Problem 4.1.4 In Eq. **??** for I_+ in the optically thick limit, we neglected the transmitted flux from the ground and also assumed that

$$\int_0^{\tau} \mathfrak{T}(\tau,\tau')d\tau' \approx \int_0^{\infty} \mathfrak{T}(\tau,\tau')d\tau' = 1 \tag{4.3}$$

Both approximations break down when τ is within a few optical depths of the ground. Find a correction to the expression for I_+ that remains valid near the ground. Show that this term yields the low-level behavior of the net flux $I_+ - I_-$ seen in the numerically computed grey gas results shown in the optically thick panels of Figure **??**.

Show that there is a similar correction term for I_- at the top of the atmosphere, and discuss its effect for a grey gas when $T(p)$ is on the dry adiabat all the way to $p = 0$. How would your answer change if T asymptoted to a constant at the top of the atmosphere?

Problem 4.1.5 *Grey gas runaway greenhouse*

Generalize formula **??** to the pressure-broadened case, i.e. the case in which the absorption coefficient increases linearly with pressure. Specifically, assume $\kappa = \kappa_o \cdot (p/p_o)$, where p_o is the reference pressure at which the absorption takes on the value $kappa_o$. Derive an expression for the coefficient $A(L/R)$ in the pressure-broadened case, and determine its value for water vapor, CO_2 and CH_4 using a numerical quadrature.

4.1.2 Pure Radiative Equilibrium

Problem 4.1.6 A solar-transparent stratosphere in a grey-gas atmosphere is in pure radiative equilibrium with upward infrared flux $I_{+,trop}$ upwelling through the tropopause. Using Eq. **??** and the fact that $I_+ - I_-$ is constant in the stratosphere, show that at the tropopause

$$I_+ + I_- = 2\frac{1+\Delta\tau}{2+\Delta\tau} \tag{4.4}$$

where $\Delta\tau \equiv \tau_\infty - \tau_{trop}$. Use this to find the stratospheric temperature at the top of the atmosphere and at the tropopause. Show that both reduce to the skin temperature when the stratosphere is optically thin.

Problem 4.1.7 *Optically thick limit*

In Section **??** it was shown that radiative transfer acts like a diffusion at wavenumbers for which the atmosphere is optically thick, and the expression for the the radiative heating H_ν was given in Eq. **??**. Suppose that the atmosphere is optically thick throughout the infrared. Show that the total infrared heating $\bar{H} \equiv \int_{ir} H_\nu d\nu$ still has the form of a diffusion, and write down the expression for the effective diffusivity $\bar{D}(T,p)$ as an integral. Radiative equilibrium in the absence of solar absorption is defined by $\bar{H} = 0$. Derive a differential equation describing the $T(p)$ which satisfies radiative equilibrium. The boundary condition for this equation is that the net infrared flux out of the top of the atmosphere is equal to the OLR (which is taken as a specified constant in this problem). Express this boundary condition as a requirement on dT/dp at $p = 0$. Note that because of the boundary condition $T = const$ is not the solution unless $OLR = 0$.

Show that this solution reduces to Eq. **??** for a grey gas with κ independent of p. Show that the equivalence continues to hold even if $\kappa = \kappa(p)$.

Problem 4.1.8 Using the results of Problem 4.1.7, compute the radiative diffusivity and the infrared radiative equlibrium solution for the following cases: (a) The atmosphere has constant absorption coefficient κ_0 within a narrow band of frequencies $|\nu - \nu_0| < \frac{1}{2}\Delta_0$, and is transparent elsewhere; (b) In addition to the absorption described in Part (a), the atmosphere has an absorption coefficient $\kappa_1 < \kappa_0$ for $\frac{1}{2}\Delta_0 < |\nu - \nu_0| < \frac{1}{2}\Delta_1$.

In both cases you may assume that the absorption coefficient is independent of pressure and temperature, that the band is narrow enough that the Planck function can be considered independent of frequency within the band, and that the atmosphere is optically thick throughout the band where the absorption coefficient is nonzero. With a bit of cleverness, Part (a) can be done analytically, but for Part (b) you will probably need to find the approximate solutions to the differential equation describing $T(p)$ using numerical methods.

Fix Δ_0 at a value corresponding to a band width of $25cm^{-1}$, and set $\Delta_1 = 2\Delta_0$ Discuss how the solution depends on ν_0, the specified OLR, and κ_1/κ_0.

Problem 4.1.9 *Muclear Winter*

"Nuclear Winter" refers to the climate following an event such as (perhaps) a global thermonuclear war, which lofts a high-altitude long-lived soot layer into the atmosphere. As an idealized form of this phenomenon, find the radiative-equilibrium temperature profile of a grey-gas atmosphere below a soot cloud which is so thick that it completely absorbs the solar radiation incident upon it. You may assume that the soot cloud is isothermal, and acts like a blackbody in the infrared. Do you expect a troposphere to form in this situation? How does the surface temperature compare to what it would be without the soot cloud? What do you think would happen to the precipitation?

Problem 4.1.10 Suppose that all the solar absorbers in an atmosphere are concentrated near the ground. This is typical of the case where the absorption is due to dust or water vapor. Specifically, assume that the profile of solar flux is $F_\circledast = (1-\alpha)S \cdot (1 - a\exp(-b\tau))$, where a and b are constants with $0 < a < 1$ and $b > 0$. Assuming that the atmosphere behaves like a grey gas in the infrared with pressure-independent κ, compute the radiative equilibrium $T(\tau)$ and the radiative equilibrium ground temperature T_g. Under what circumstances does the atmosphere become internally unstable near the ground? How does solar absorption affect the unstable temperature jump between the ground and the overlying air? To keep the problem simple, you may assume that the ground has zero albedo.

Problem 4.1.11 Consider a grey-gas atmosphere with a sharply defined internal absorbing layer at pressure p_a. The profile of downward solar flux is $F_{\circledast} = (1-\alpha)S \cdot \frac{1}{2}(1+\tanh((p_a-p)/\Delta p))$, where Δp is the thickness of the absorbing layer. Find an expression for the radiative equilibrium temperature profile, evaluate it numerically, and plot it. Determine analytically where the absorbing layer causes the temperature to increase with height and where it causes the temperature to decrease with height. Under what circumstances does the absorbing layer make the atmosphere unstable to convection? You may assume that the absorption coefficient is independent of pressure in this problem.

Problem 4.1.12 *Global warming and Stratospheric cooling*

One of the signatures of anthropogenic global warming is that, in response to increases in greenhouse gas concentrations, the stratosphere cools at the same time the troposphere warms. This problem explores the phenomenon for a grey gas atmosphere.

Suppose that an atmosphere acts as an ideal grey gas in the infrared with an absorption coefficient κ independent of both pressure and temperature. Suppose that *as a function of pressure* the solar flux decays according to the law $F_{\circledast} = S\exp -bp/p_s$ for some constant b. To keep things simple, you may assume that the albedo of the ground is zero, so that all the incident solar radiation is absorbed either in the atmosphere or at the ground. Thus, you can assume that $OLR = S$, as in the derivation of Eq. **??**. This also means you need not worry about absorption of upwelling solar radiation reflected from the ground.

Show that in the notation of Section **??**, $\tau_\infty = \kappa p_s/g$, $\tau_\infty - \tau = \kappa p/g$ and $\tau_S = \tau_\infty/b$. With these relations you can find the profile $T(p)$ using the solution given in Eq. **??**. Since atmospheric observations are made using pressure rather than τ as the vertical coordinates, it is important to plot the temperature as a function of pressure if one wants to relate the theoretical behavior to observations.

Increasing the greenhouse gas concentration while keeping the distribution of solar absorbers fixed is equivalent to increasing τ_∞ while holding b fixed. On the same graph, plot $T(p)$ for $\tau_\infty =$ .25,.5,1,1.5,2 and 4, for a fixed value of b. Make graphs of this type for a range of b and explore the behavior as b ranges from very small values (e.g .01) where there is essentially no solar absorption to very large values (e.g 10) where all the solar energy is absorbed in the upper atmosphere. Under what circumstances does an increase of the greenhouse gas concentration warm the lower atmosphere but cool the upper atmosphere?

4.1.3 Real Gas basics

Problem 4.1.13 Derive the result given in Eq. **??**. To do this, substitute $y = 1/\zeta$ and then do an integration by parts. You will need the well-known result for the integral of the Gaussian, $\int_0^\infty \exp(-y^2)dy = \sqrt{\pi}/2$.

Problem 4.1.14 Discuss the behavior of the function $F(\ell) = \int \kappa_m^{-1} P(\kappa/\kappa_m)\exp(-\kappa\ell)d\kappa$ for: (a) $P = 2(1-x)$; (b) $P = 2x$; (c) $P = 6x(1-x)$. (d) $P = 4\max(0, 2x-1)$ In all cases assume $P = 0$ unless

$0 \leq \kappa \leq \kappa_m$. Pay particular attention to the behavior of F at large ℓ. Under what circumstances is the decay slower than exponential? What determines the decay rate at large ℓ?

Show that if $P(x) \sim x^b$ at small x, then $F(\ell)$ decays like $\ell^{-(b+1)}$ at large ℓ.

Problem 4.1.15 Consider a gas which is transparent outside of a single frequency band, within which the transmission function is described by the Malkmus model. Compute the radiative cooling rate for an unbounded atmosphere which has a temperature discontinuity at pressure p_o, but is otherwise isothermal. Show that in the limit where the strong-line behavior dominates, there is a square-root singularity in the vicinity of the discontinuity, which tends to wipe out the temperature jump. Generalize this result by computing the radiative diffusivity for the one-band Malkmus model, in the optically thick limit.

Problem 4.1.16 Write a transmission function that uses the exponential-sum representation to compute the transmission function as a function of path, for a single band for which exponential sum coefficients have been tabulated. You may assume that the gas has uniform temperature along the path. Read in exponential sum data for a few bands near the 666 cm^{-1} principle CO_2 absorption feature, and graph the transmission as a function of path.

Exponential sum tables for various gases are found in the subdirectory `ExpSumTables` of the Workbook datasets directory for this chapter, available as part of the online supplement.

`Python Tips:` It is easier to organize the calculation if you define a `bandData` object which stores all the necessary information for each band. Then, when you need to sum up over multiple bands, you only need to make a list of bands and loop over the `bandData` objects in the list. This is how the homebrew radiation code is implemented in the courseware.

Python Tips: You can compute your own exponential sum tables using the function `makeEsumTable` in the chapter script `PyTran.py`, which makes use of the `HITRAN` database. A limited subset of `HITRAN` is included in the Workbook dataset directory for this chapter.

Problem 4.1.17 *Working with HITRAN*

A subset of the `HITRAN` database and its documentation is included in the Workbook datasets directory for this chapter, in the subdirectory `hitran`. The database itself, separated into files for each molecule, is in `hitran/ByMolecule`. The data is in plain text table format, and is not very hard to read. Read the documentation on the data format, and write a routine that reads in line strengths and line widths for CO_2. These values will be at the standard pressure and temperature for the database. To get a feel for the data, compute and plot the number of lines present in a series of bins of width $50cm^{-1}$, covering the wavenumber range for which data is available. Compute and plot the median, maximum and minimum line strength in each bin.

Then, assuming the Lorentz line shape, use the line strength, position and width information to sum the contributions of each line (at standard width) and create a plot of the absorption coefficient as a function of wavenumber. When summing the contributions, you should cut off the contribution past a certain threshold distance from the line center; there are various empirical far-tail line shapes in use, but for this problem a simple truncation will do. Try truncating at a fixed number of line-widths (e.g. 100). Try truncating instead at a fixed wavenumber distance (e.g. 10 cm^{-1} from the line center). How much does the result depend on the cutoff prescription? These calculations will take quite a lot of computer time, even if you are quite clever at writing an efficient algorithm. You can keep down the time needed by restricting attention to just a portion of the wavenumber domain.

Next, fix the line cutoff prescription and look at how the absorption spectrum changes as you increase the pressure from 100 mb to $100 bar$. You do this by rescaling line widths and peak values according to pressure, as described in the text. Note that the curves become smoother at higher pressure. Why is this? Compare results with a cutoff based on a fixed number of widths with those based on a fixed wavenumber value.

Once you've mastered these techniques, the `HITRAN` database can provide endless hours of enjoyable perusal of spectroscopic properties. Take a look at some infrared absorption properties for molecules not discussed in the text. Some particularly interesting ones to look at are ozone, SO_2 and the minor isotopologues of CO_2.

Python tips: The chapter script `PyTran.py` reads the line database, computes absorption coefficients, and provides functions to perform utility tasks such as generation of exponential-sum tables.

4.1.4 Calculations with polynomial OLR fits

The following problems involve calculations with polynomial fits to the `ccm` model OLR calculation for an Earthlike atmosphere. The fitting coefficients needed are found in Tables **??** and **??** in the text.

Python Tips: The chapter script `OLRPoly.py` contains the fitting coefficients given in the tables, and also contains an interpolation routine that makes it easy to compute $OLR(T_g, CO_2)$ for arbitrary values of the arguments.

Problem 4.1.18 For a dry air Earth atmosphere on the dry adiabat, with $1000 ppmv$ of CO_2 mixed in, the $OLR(T_g)$ computed from the `ccm` radiation model can be well fit by a polynomial $a+bx+cx^2$, where $x \equiv T_s - 275.$. The coefficients $a = 273.9$,$b = 3.9436$ and $c = .026987$ provide a fit good to better than 1% over the temperature range $250K$ to $350K$.

Compute and graph the radiating temperature T_{rad} and radiating pressure p_{rad} as a function of T_g. Also, make a graph comparing $dOLR/dT_g$ for the polynomial fit with the grey gas value $4\sigma T_g^3$. Why does the grey gas estimate differ from the actual slope? (Recall that the slope is important because it provides a measure of climate sensitivity).

Problem 4.1.19 Table **??** gives a cubic polynomial fit of OLR vs. T_g for a moist Earth atmosphere with fixed relative humidity, with temperature on the moist adiabat. Results are given for several different values of CO_2 concentration.

Compute and graph the radiating temperature T_{rad} and radiating pressure p_{rad} as a function of T_g. Note that since the OLR fit was done for an atmosphere on the moist adiabat, you will need to use the moist adiabat to compute p_{rad} from T_{rad}. Also, make a graph comparing $dOLR/dT_g$ for the polynomial fit with the grey gas value $4\sigma T_g^3$. Why does the grey gas estimate differ from the actual slope? Discuss how the temperature affects the comparison. How does $dOLR/dT_g$ compare with the dry value you obtained in Problem 4.1.18? (Recall that the slope is important because it provides a measure of climate sensitivity). Carry out your analysis for several different values of CO_2 concentration.

Python tips: Use the object `MoistAdiabat` in the `phys` courseware module to help you compute p_{rad}.

Problem 4.1.20 Recompute the ice-albedo bifurcation diagram in Fig. **??** using the polynomial

OLR fit for 50% relative humidity. Do the calculation for several different values of CO_2 concentration, and discuss the results in light of Earth's actual climate history. *Note:* This calculation neglects the effect of clouds, so for any given $L_\circledast$ the temperature in the unglaciated state will tend to be too warm and the temperature in the fully glaciated state will tend to be too cold.

Problem 4.1.21 Recompute the ice-albedo bifurcation diagrams in Fig. **??** using the polynomial OLR fit for 50% relative humidity. In this case, use the CO_2 concentration as your control parameter instead of p_{rad}. Show results for various different values of $L_\circledast$ ranging from 500 W/m^2 to 3000 W/m^2. In cases where the CO_2 concentration you need to fill out the diagram lies outside the range of validity of the polynomial fit, just show the portion of the diagram for which the fit is valid and sketch in the qualitative behavior of the missing bits of the curve. Note that the remark about clouds in Problem 4.1.20 applies to this problem as well.

Hint: As was the case for the bifurcation diagram using $L_\circledast$ as the bifurcation parameter, it is easiest to find the control parameter which yields a given temperature T_s and then plot the control parameter (CO_2 in this case against T_s sideways. This procedure is easier because there is a unique CO_2 corresponding to each T_s, whereas there are multiple solutions for T_s for a given CO_2. To find the CO_2 corresponding to T_s, first find the OLR needed to achieve that temperature, then use a Newton's method iteration to find the value of $\ln CO_2$ that corresponds to this OLR. You will need a function that gives you $OLR(T_s, \ln CO_2)$; this is obtained by interpolating the coefficients in Table **??** to the desired value of $\ln CO_2$ and then evaluating the resulting polynomial at T_s.

Problem 4.1.22 Consider an Earthlike planet orbiting a star with luminosity such that the solar constant is 2000 W/m^2 at a distance of 1 *a.u.* from the star. Using the information in Table **??**, find the CO_2 needed to maintain an Earthlike habitable temperature as a function of distance from the star. Do this for various assumptions about the humidity of the atmosphere. You may assume the albedo of the planet to be 20%.

4.1.5 Real Gas and Semi-grey Radiation Computations

Many of the problems in this and the next section require use of the homebrew exponential sums radiation code, though the more Earthlike ones can be done using the `ccm` radiation code as an alternative. The basic building block of the homebrew code is a routine to compute the transmission function for a single band, which was done in Problem 4.1.16. All the necessary exponential sum tables have been precomputed for you, and are found in the workbook dataset directory for this chapter, available in the online supplement. The tables are in the subdirectory `ExpSumTables`. The rest of the procedure is described in the text. It is not too hard, but still it takes some doing to implement. Hopefully, if you are not using the ready-made `Python` courseware, your instructor will have done some of the work for you in advance, in whatever programming environment is being used as an alternative.

Python tip: The homebrew radiation computation is implemented in the modules `miniClimt.py` for a well-mixed greenhouse gas and in `miniClimtFancy.py` for an inhomogeneous greenhouse gas. The latter also implements a crude representation of temperature scaling of line strengths, and an allowance for the difference between self-broadened and foreign-broadened absorption. The `ccm` radiation model has also been made available through a user-friendly `Python` interface, using the `climt` module to interface to the fast, compiled `Fortran` code. Some functions useful for radiation calculations using the `ccm` model are defined in the chapter script `ccmradFunctions`.

Problem 4.1.23 The semigrey skin temperature is defined by Eq. **??**. Using Newton's method or some similar root-finder, make a plot of T_{skin}/T_g as a function of $h\nu_o/kT_g$. Analytically derive the behavior of T_{skin}/T_g in the limit of small and large $h\nu_o/kT_g$.

Problem 4.1.24 Numerically determine the pure-radiative equilibrium for a semigrey atmosphere, in equilibrium with a specified ground temperature T_g. You find the equilibrium by numerically computing the net radiative cooling as an integral (you should already have code to do this from having done the previous problems), and then time-stepping the temperature equation until equilibrium is reached.

Let ν_o be the frequency at the center of the narrow absorbing band of the atmosphere. Pick a nu_o in the infrared range, and explore how the equilibrium profile $T(p)/T_g$ varies as you change T_g. How does the result depend on nu_o?

Python Tips: The numerical computation can be done using a variation of the general radiative-convective equilibrium script `RadConvEq.py`. To do the semigrey case, just edit the `radmodel.bandParams` line so that the one-band Oobleck data is used in the homebrew radiation model.

Problem 4.1.25 This problem addresses the role of the water vapor continua in determining the Kombayashi-Ingersoll limit. Consider a pure saturated water vapor atmosphere, and compute the OLR as a function of surface temperature, neglecting all absorption due to the two water vapor continua. In other words, recompute Fig. **??** without the continua. Then, re-do your calculation with only the lower wavenumber continuum region included. Finally, compute the OLR including the higher wavenumber continuum region, but assuming the atmosphere emits no radiation at wavenumbers outside the continuum. How well does this calculation reproduce the Kombayashi-Ingersoll limit?

Python Tips: Using the homebrew radiation code as in the script `RealGasRunaway.py`, you can suppress the continuum by simply setting the continuum function to `NoContinuum`. To suppress just the higher wavenumber continuum, you need to go into the homebrew radiation module imported by `RealGasRunaway.py` and edit the water vapor continuum function so it only calculates the lower wavenumber continuum. To do the last part of the problem, you need to edit the radiation code so that it sums up only contributions to OLR from the continuum regions, and ignores contributions from all other wavenumbers.

Problem 4.1.26 Using the homebrew radiation model for a saturated pure water vapor atmosphere (as in Fig. **??**), make a graph of the OLR as a function of surface gravity g for a fixed surface temperature $T_g = 400K$, and demonstrate the fit of Eq. **??** for the parameters given in the text.

4.1.6 Tropopause height

Problem 4.1.27 This problem examines the effect of lapse rate on the tropopause height for a real gas.

Consider an atmosphere whose temperature profile is governed by the moist adiabat for a mixture of water vapor in air, but whose radiative properties are determined by a mixture of 300 *ppmv* of CO_2 in dry air. In other words, in this problem we include the effect of water vapor on the adiabat, but neglect the radiative effects of water vapor. Using hard-convective adjustment, compute the tropopause height by running a radiative-convective model to equilibrium. Carry out the calculation for surface temperatures of $280K$ and $300K$. Compare the results with the what you would get for these two temperatures using the dry air adiabat instead of the moist adiabat.

Note: You can spot the tropopause height either by noting where the temperature profile departs from the adiabat, or by looking for where the radiative heating falls to essentially zero.

Python Tips: This problem is done by making use of the script `RadConvEq.py`, uncommenting the options for enabling convective adjustment and for using the moist adiabat.

Problem 4.1.28 *Grey Gas Tropopause Height vs.* τ_∞.

In this problem, you will find the tropopause height for a grey-gas atmosphere in radiative-convective equilibrium. In the troposphere, the temperature profile is assumed to be on the ideal gas dry adiabat. The low level air temperature is assumed to be identical to the ground temperature. The atmosphere is transparent to solar radiation. We'll solve the problem by specifying the ground temperature T_g and finding the atmosphere which is in equilibrium with this value.

You do not need to solve this problem by time-stepping. It still needs to be done with some numerical analysis, but the problem can be solved in terms of numerical quadrature and a Newton's method root finder. The solution scheme runs as follows. First, re-write the problem in terms of the optical thickness coordinate τ. τ_∞ then becomes a specified parameter of the problem. Suppose that the tropopause is located at τ_{trop}. From the dry adiabat, we then know $T(\tau)$ between the tropopause and the ground, whence the upward flux $I_{+,trop}$ can be computed as a definite integral. This is the same integral as used in the grey all-troposphere model in the text, except that it is only carried out only to τ_{trop} and not all the way to $\tau = 0$. You will need to implement a function that evaluates this integral using numerical quadrature. This is the upward flux which heats the stratosphere, since there is no solar absorption. The temperature for a stratosphere illuminated by upward flux I_+ was determined in Problem 4.1.6.

By evaluating this at the tropopause, you have the temperature $T_{strat}(\tau_{trop}$. In general, this will be discontinuous with the troposphere temperature just below the tropopause. If the tropopause is too low, then the temperature jump is statically unstable, so the troposphere will mix upwards by convection. The usual physical assumption governing the tropause height is that the mixing proceeds until the unstable jump is just eliminated. Thus, the condition we impose at the tropopause is that the temperature be continuous there. This uniquely determines the tropopause height. The numerical calculation thus consists of three parts: a numerical quadrature routine to compute $I_{+,trop}$, a fairly trivial calculation of the corresponding stratospheric temperature, and a Newton's method adjustment of τ_{trop} until the temperature continuity condition is satisfied.

Write down the equations which embody the temperature continuity condition, and show that T_g drops out of the problem if all temperatures are written as T/T_g. In other words, the value of τ_{trop} which satisfies the continuity condition is independent of T_g.

Using the above procedure, make a graph of how the tropopause height depends on τ_∞ for various values of R/c_p. Can you explain the behavior in the optically thin limit analytically? Show that without pressure broadening (i.e. constant κ) the tropopause moves to the top of the atmosphere in the optically thick limit if $4R/c_p > 1$, and moves to the ground in the optically thick limit when $4R/c_p < 1$. Note that this result says that, in circumstances when the radiative equilibrium is statically stable at all levels, the atmosphere becomes all-stratosphere in the optically thick limit. How does the tropopause behave if you assume linear pressure broadening, so $\kappa = \kappa_o \cdot (p/p_o)$?

If you are very good at evaluating definite integrals asymptotically by steepest descent, you can actually do the optically thick limit analytically.

Problem 4.1.29 *Real Gas Tropopause Height vs.* τ_∞.

Consider a solar-transparent dry atmosphere consisting of a mixture of Earth air with a mass concentration q of CO_2. The troposphere is assumed to be on the dry adiabat. The low level air temperature is identical to the ground temperature. As in the text, we specify the ground temperature T_g and find the atmosphere which is in infrared radiative-convective equilibrium with the ground. Once the infrared cooling profile is computed from a radiation model, the equilibrium is found by time-stepping, applying hard convective adjustment to eliminate unstable layers, as in the examples done in the text. Any radiation model will do, but it is suggested that you use the homebrew exponential sum model. This problem requires a fairly large amount of computation time, so it is most suitable as an extended project spread over a few weeks. You can keep the computation time down by using only 20-30 points in the vertical, which should be sufficient to reproduce the basic behavior.

Take $T_g = 300K$ and $p_s = 10^5 Pa$, and explore the behavior of tropopause height as q ranges from small values to unity. Note that you ought to change the value of R/c_p as q changes, but to keep things simple you may keep R/c_p constant at the value $\frac{2}{7}$ appropriate to dry Earth air. Next, explore how the behavior depends on T_g, for a few values ranging from $200K$ to $400K$. The cold case has some features in common with Snowball Earth.

Now re-do the problem for a pure CO_2 atmosphere on a CO_2 dry adiabat. In this case, you vary the optical thickness by varying the surface pressure p_s. Keeping $T_g = 300K$, try $p_s = 1000Pa$ (similar to Mars), $p_s = 2bar$, $p_s = 20bar$, and $p_s = 90bar$. The last of these is in the thick-atmosphere regime of Venus; recompute it at $T_g = 700K$. How does the tropopause height change?

Note that for the hot, high-CO_2 case you will need to use an exponential sum table that goes far enough out in wavenumber to capture the surface emission. If you need to, you may assume that the atmosphere is so optically thick in the wavenumbers past the end of the table that the fluxes there do not contribute significantly to the infrared cooling.

There are various other variants of this problem you can try. For example, you can do a problem with water vapor in air on the moist adiabat; in this case you would fix the relative humidity, and the optical thickness would increase as you went to larger temperatures, which yield higher water contents. Alternately, using the `ccm` radiation code you can do an atmosphere consisting of water vapor, air and CO_2, and vary the optical thickness by varying the CO_2 with fixed T_g.

Python tip: This calculation is implemented in the chapter script `RadConvEq.py`, which makes use of the homebrew exponential sums radiation code.

Chapter 5

Scattering problems

5.1 Workbook

5.1.1 Scattering basics

Problem 5.1.1 An atmosphere consists of a mixture of scattering particles with a grey gas having pressure-dependent absorption coefficient $\kappa(p) = \kappa_o \cdot (p/p_o)$. The mass concentration of scatterers is constant, and the scatterers have scattering efficiency $Q_{sca} = 2$. The scatterers are spherical and have the density of water. They do not absorb. Compute the profile of optical thickness $\tau(p)$ and also the profile of single-scattering albedo $\omega_o(p)$. Tabulate τ and ω_o on a discrete grid of pressure values. Using these arrays, make use of polynomial interpolation to write a function that returns $\omega_o(\tau)$.

Python tips: The interpolation is easily done using the `interp` class.

Problem 5.1.2 *Partial reflection at an interface*

When a beam of radiation hits the interface of a smooth transparent body such as an ocean, lake, or clear ice, part of the beam is transmitted but part is reflected. Partial reflection of this sort is important in determining the albedo of such surfaces, most commonly for liquid bodies. The albedo for such surfaces depends rather strongly on the angle of incidence of the beam. Maxwell's Equations can easily be solved to yield the reflection at a discontinuity in index of refraction. For the case where the index of refraction is purely real, the answer is given by *Fresnel's formula.* For an unpolarized beam of light hitting a smooth surface with angle Θ_i relative to the normal, the albedo is $\alpha = \frac{1}{2}(R_s + R_p)$, where

$$R_s = \left(\frac{\sin(\Theta_t - \Theta_i)}{\sin(\Theta_t + \Theta_i)}\right)^2, R_p = \left(\frac{\tan(\Theta_t - \Theta_i)}{\tan(\Theta_t + \Theta_i)}\right)^2 \tag{5.1}$$

and Θ_t is the angle of the transmitted (i.e. refracted) beam relative to the normal, given by Snell's law. For a horizontal surface such as an ocean or lake, Θ_i is the zenith angle. Stellar illumination can generally be considered unpolarized.

Using Snell's law, compute the albedo for near-normal incidence – i.e.$\Theta_i \approx \Theta_t \approx 0$ – as a function of the jump in index of refraction. Compute the albedo for visible radiation at $0.6\mu m$ striking a liquid methane lake at normal incidence, and at an angle of 45^o. Do the same for a liquid water lake. How different are the results for $UV - B$?

Problem 5.1.3 Compute the Rayleigh scattering optical thickness for Titan's atmosphere, idealizing it as consisting of 1.5 *bar* of pure N_2. Do the calculation for visible, near-infrared and UV-B wavelengths. If Titan's stratospheric haze clouds were taken away, would Rayleigh scattering contribute significantly to Titan's albedo?

Problem 5.1.4 This problem involves computation of the properties of Mie scattering. The computation requires the summing of a complicated infinite series whose terms require evaluation of Bessel functions. You are not expected to program this summation yourself; Mie summation routines are widely available for many programming languages, and a few are provided as part of the online software supplement.

Re-do Fig. **??** for a range of different real indices of refraction $n_R > 1$. (Note that $n_R = 1.3$ for liquid water).

Python tips: The Mie series summation is provided in the chapter script `Mie.py`.

Problem 5.1.5 The file `co2i4a.rfi.txt` in the Workbook datasets directory for this chapter contains the real and imaginary index of refraction for CO_2 ice as a function of wavelength. Assuming spherical CO_2 ice particles with a radius of 5 μm, use the Mie scattering solution to determine the scattering efficiency, absorption efficiency and asymmetry factor as a function of wavelength, from the thermal infrared through the visible range. Plot the results. Also, save them in a file, since you will need them in Problem 5.1.17 below.

Python tips: Modify `Mie.py` so that it reads in the data and loops over wavelength.

5.1.2 Use and derivation of basic 2-stream solutions

Problem 5.1.6 Consider a planet with a 5 *bar* atmosphere of N_2 under a surface gravity twice that of Earth. The surface of the planet is perfectly absorbing. There are no clouds, and you may ignore absorption of stellar radiation within the atmosphere. Compute the Rayleigh scattering albedo this planet would have in orbit around stars of spectral class M,G and A. How does your answer change if the atmosphere consists of CO_2 instead? (Note that in that case the neglect of stellar absorption is less justifiable).

Problem 5.1.7 Doubling the Earth's pre-industrial CO_2 concentration would reduce the OLR by about 4 W/m^2, requiring the atmosphere to warm up to restore balance. Suppose instead we decide to increase the albedo so as to reduce the absorbed solar radiation by 4 W/m^2. How much increase of albedo does this require? Assuming stratospheric sulfate aerosols to stay aloft for just one year, how much mass of SO_2 would have to be injected into the stratosphere annually to achieve this albedo increase? Assume that the aerosols form into droplets with a radius of 1 μm, and that each SO_2 molecule injected forms one H_2SO_4 molecule, which forms a droplet which is composed of half water and half sulfuric acid.

Note: Don't try this at home! It will put your planet in a very precarious state because the extra CO_2 stays in the atmosphere around a millennium, whereas the sulfate aerosols will fall out in a few years. The planet is thus at risk of being hit with a century's worth of global warming in just a few years, if the aerosol injection should ever cease.

Problem 5.1.8 Derive an expression for the albedo of a planet for which the incoming radiation is diffuse and there is no incoming direct beam radiation. Assume the ground to be perfectly absorbing ($\alpha_g = 0$) and the atmosphere to contain only conservative scatterers ($\omega_o = 1$). The derivation is similar to that used to obtain the albedo for direct-beam incoming radiation, except that the direct beam source terms are dropped and I_- is no longer assumed to vanish at the top of the atmosphere, but rather takes on a specified value $I_{-,\infty}$ which defines the incoming flux. Show that the expression is the expression for α'_a given in Eq **??**, and state why the diffuse albedo of the atmosphere is the same regardless of whether radiation is coming in from above or bouncing off the atmosphere from below.

Use the expression for the diffuse albedo to derive Eq **??** for the conservative infrared-scattering greenhouse effect. *Hint:* For conservative scattering, the transmission through the atmosphere is $1 - \alpha'_a$ times the incident flux.

Problem 5.1.9 Solve Eq. **??** for $I_+(0)$ and discuss the result. Using this as the bottom boundary condition for the conservative-scattering two-stream equations with direct beam incoming radiation, derive Eq. **??** for the planetary albedo allowing for surface reflection. The derivation is precisely the same as the derivation used to obtain Eq. **??**, except for the change in the boundary condition on $I_+(0)$. Note that it is not necessary to re-do the derivation from scratch. Since the problem is linear, it suffices to add the portion of I_+ transmitted through the atmosphere to the previous expression for $I_{+,\infty}$, and use the result to obtain the modification to the albedo formula.

Problem 5.1.10 For a conservatively scattering atmosphere over a perfectly absorbing surface, plot the albedo as a function of the zenith angle, for an optically thick case with $\tau_\infty = 10$. Show results for various values of the asymmetry factor, and compare results between the hemi-isotropic, quadrature and Eddington approximations. When do you get spurious negative values of the albedo? Derive an analytical criterion which defines the circumstances under which the albedo goes negative.

Problem 5.1.11 For a conservatively scattering atmosphere over a perfectly absorbing surface, plot the albedo as a function of the optical depth for a few different values of the zenith angle. Fix the zenith angle at 45^o, and do the same for a range of values of the asymmetry parameter. Discuss the exponential and algebraic stages of increase of albedo with optical thickness. For this problem, you may restrict attention to the hemi-isotropic approximation.

Problem 5.1.12 Generalize the albedo formula in Eq. **??** to allow for finite optical depth of the atmosphere. To keep things simple, you may assume that the asymmetry factor is zero and that the ground is perfectly absorbing.

Problem 5.1.13 *A Diabolical Mirror*

Suppose you could make a mirror that is perfectly transparent to radiation with wavenumbers greater than 5000 cm^{-1} but perfectly reflective at lower wavenumbers. (Optical supply companies actually do sell mirrors that are rather like this.) You put this mirror atop a well-insulated box that does not allow any heat to leak out, so that the only energy that leaves the box is in the form of radiation that escapes through the mirror. The interior of the box has unit emissivity at all wavenumbers.

You expose this box to sunlight having a Planck spectrum corresponding to a temperature of 6000 K. Once the box reaches equilibrium, what is its temperature? Does the equilibrium temperature depend on the value of the radiation *flux* incident on the box – i.e. does it matter whether you do this experiment on Mercury or Mars?

Problem 5.1.14 Consider a planet with a pure N_2 atmosphere and a surface that is perfectly absorbing throughout the spectrum. High N_2 ice clouds form, which have zero infrared emissivity, but are highly reflective in the thermal infrared. The albedo in the thermal infrared is α_{ir}, and the albedo in the solar spectrum which illuminates the planet is α_{sw}. What is the equilibrium temperature of the planet? You may ignore gaseous Rayleigh scattering, and assume that the atmosphere is purely transparent in the infrared.

Problem 5.1.15 In the text Eq. **??**, describing the scattering greenhouse effect in the presence of absorption and emission, was derived assuming the atmosphere to be optically thick. Generalize this result to the case where the atmosphere is not assumed to be optically thick. Also, allow the ground temperature T_g to be different from the temperature T of the atmosphere, which should still be assumed isothermal. Write your solution in terms of an atmosphereic transmission coefficient A_{trans} and an atmospheric emission coefficient A_{emiss}, so that the OLR takes the form $A_{trans}\pi B(\nu, T_g) + A_{emiss}\pi B(\nu, T_g)$. Show that your result reduces to the previous one in the optically thick limit. Discuss the behavior in the limit $\omega_o \rightarrow 1$ with fixed (but large) τ_∞^*.

Note that this solution applies as well when the absorbing/scattering layer is a high isothermal cloud deck and there is a layer of transparent non-scattering atmosphere between the ground and the cloud base.

Problem 5.1.16 *Net effect of Venus clouds*

Assume that the clouds of Venus are made of spherical concentrated sulfuric acid droplets with a radius of 5 μm. How much mass of sulfuric acid per square meter must be present in the cloud deck in order to give Venus an albedo of 70% in the visible spectrum at 0.5 μm? You will need to make use of conservative Mie scattering results corresponding to the index of refraction of sulfuric acid;remember to include the effect of the asymmetry factor in your albedo calculation. You may ignore all absorption in the cloud deck, and assume any radiation that passes through is absorbed in the atmosphere or at the surface. (In reality some would be bounced back by Rayleigh scattering.)

Now let's estimate the effect of the same cloud deck on OLR. Venus clouds are in a high cloud deck near the top of the atmosphere, and can be thought of as a partially absorbing infrared mirror acting as a leaky lid. At some infrared wavelengths, Venus clouds are strongly absorbing and emitting, and don't scatter significantly. At others, scattering is significant. As an idealization, compute the diffuse infrared albedo of the cloud deck at wavelengths of 10 μm and 2 μm, assuming the clouds to be conservative scatterers. Compare these with the visible albedo you estimated previously. Could the scattering greenhouse effect significantly offset the high visible albedo? At the opposite extreme, assume that the clouds are so strongly absorbing that the cloud deck acts like an ideal blackbody radiating at temperature T_{cloud}. What value would this temperature have to have in order for the emission to balance the absorbed solar radiation, given the visible albedo assumed above? Compare this to the observed temperature range of the cloud deck, which is $230K$-$260K$. Could the absorption/emission greenhouse effect also significantly offset the high visible albedo?

Finally, let's get an idea of where in the infrared spectrum the clouds are acting as absorbers and where they are acting as scatterers. At 10 μm the index of refraction is $(n_R, n_I) = (1.9, .46)$, and at 2 μm it is $(1.4, .0048)$. Use Mie theory to determine the scattering and absorption efficiencies, and hence the single scattering albedo ω_o for the cloud deck. You will also need the asymmetry factors. Then plug the result into Eq. **??**, which describes the emission from a thick, isothermal cloud deck. When does the layer radiate almost like a blackbody? When does the scattering significantly reduce the emission? Note in circumstances when the scattering significantly reduces the emission, the cloud layer will also be quite reflective to infrared hitting the cloud deck *from below*. You can get a more quantitatitive appreciation of this by examining the solution to Problem 5.1.15.

Problem 5.1.17 Radiative effects of CO_2 ice clouds

This lengthy (but hopefully rewarding) problem serves as an archetype for the determination of the net climate effect of high clouds which both scatter and absorb. The problem was originally motivated by the effect of CO_2 ice clouds on Early Mars, but the occurrence of such clouds is generic to a broad class of problems involving cold, thick CO_2 atmospheres. While the specifics are worked out for CO_2 optical properties, the techniques carry over without change to other substances. The model problem described below can readily be generalized to include a nontrivial gaseous atmosphere below the cloud deck.

The model problem we consider starts with a transparent nonscattering atmosphere above a surface with temperature T_g. The surface is an ideal blackbody in the thermal infrared spectrum, but has an albedo α_g for wavelengths shorter than 1 μm. T_g is assumed to be low enough that the surface emission at near-IR and shorter wavelengths can be neglected. To this base state, we add a cloud deck at the top of the atmosphere. This will reflect back some of the incoming stellar radiation, and will also reduce the outgoing thermal radiation. The cloud deck is isothermal with temperature T_{cloud}. It is composed of idealized spherical CO_2 ice particles with a radius of 5 μm. The mass path of particles in the cloud deck is 0.05 kg/m^2. All absorption and scattering is due to the particles, and the background gas is considered transparent.

First, let's compute the albedo for incoming stellar radiation, assuming that the clouds are conservative scatterers. Use Eq. **??** together with your CO_2 ice Mie scattering results from Problem 5.1.5 to compute the albedo as a function of wavenumber. You may assume a fixed zenith angle of 45^o. Show results for $\alpha_g = 0$, $\alpha_g = .3$ and $\alpha_g = .7$. The second of these approximates the Rayleigh scattering from a thick CO_2 atmosphere, and the last is typical of a Snowball Earth case. Finally, for each case find the mean albedo weighted according to the Planck function for a G star with $T_{\circledast} = 6000K$ and for an M star with $T_{\circledast} = 3000K$. Examine the absorption efficiencies over the stellar spectrum from the Mie calculation. How valid is the assumption of conservative scattering for the G star? For the M star?

Next, we'll find the effect of the cloud on the OLR. In this case we use the absorption as well as scattering efficiencies from the Mie calculation, and do not assume conservative scattering. Use the Mie results to compute ω_o, and then plug all the results into the formula you derived in Problem 5.1.15 in order to compute the emission and transmission coefficients of the cloud deck. Plot these as a function of wavenumber, and also plot the OLR spectrum for $T_{cloud} = 150K$ and $T_g = 260k$. Compare the results with the Planck spectra corresponding to T_{cloud} and T_g. In what parts of the spectrum is the cloud acting primarily like a scatterer? In what parts is it acting like an absorber?

Finally, put in some number corresponding to Early Mars insolation. First, assume that T_g is in equilibrium with the absorbed solar radiation given $\alpha_g = .3$. Then, compute how much the solar absorption goes down when you introduce the cloud, and compare to how much the net OLR goes down. For the latter, assume the cloud temperature is $150K$. Does the cloud act to warm or cool the planet? How sensitive is your result to the assumed cloud temperature?

5.1.3 Numerical 2-stream solutions

Problem 5.1.18 One of the new numerical techniques introduced in this chapter is the solution of two-point linear boundary value problems using the ODE method, in which the solution is built up from a pair of numerically determined fundamental solutions. Use this technique to write a routine to solve $d^2Y/dx^2 - aY = F(x)$ subject to $Y(0) = Y(1) = 0$, integrating the equations all the way from one boundary to the other. Carry out the integration for $F = \sin \pi x$ and show what goes wrong when a becomes large and positive (the analog of the optically thick case described in the text). Then modify your routine to use the local-ODE method described in the text, and compare the results with the exact analytic solution.

Problem 5.1.19 For this problem you will need an implementation of the numerical solution to the two-stream equations, using the piecewise ODE method described in Section **??**. This problem only deals with the solution for a single absorption coefficient corresponding to a single frequency; a problem involving exponential sums will be done later. The implementation is not intrinsically difficult but getting all the algebra right for the matching conditions takes a bit of doing, and if you are not using the Python implementation, it is to be hoped that your instructor has done some of the groundwork for you.

First, take a case with constant T and ω_o, and check the numerical results against some of the conservative and nonconservative scattering solutions given in Section **??**. Also check the results against your answer to Problem 5.1.12. In that case, re-run the numerical solution with a nonzero asymmetry factor and see how the result changes. Extend the code so that it computes heating profiles, and take a look at some of those. (Recall that in this kind of calculation you do *not* need to compute heating from the fluxes by taking a finite difference. You can get them accurately by simply evaluating the right hand side of the differential equation governing the fluxes).

Finally, let's do something a little new with the calculation. We'll do a highly simplified estimate of the emission Titan would have at 100 cm^{-1} in the absence of its high haze clouds. Assume that the atmosphere is on the dry N_2 adiabat with surface temperature $95K$ and a surface pressure of 1.5 bar, under Titan gravity. The gaseous absorption is dominated by the N_2 continuum at this wavenumber. The absorption coefficient is $1.3 \cdot 10^{-6}$ at 100 mb and 100 K, and can be considered to scale linearly with pressure. It is also temperature dependent, but you may ignore that effect, to keep things simple. To this atmosphere, add a liquid CH_4 cloud with uniform cloud mass concentration in the lowest $500mb$ of the atmosphere. The cloud is made of spherical droplets with a radius of 10 μm. They are purely scattering, with $n_R = 1.3$. Find the OLR at 100 cm^{-1} as a function of the cloud mass path. For selected values, calculate the profile of the heating rate, and discuss how it is affected by the cloud.

Python tips: The piecewise ODE solutiom is implememted in the Chapter Script `TwoStreamScatter.py`.

Problem 5.1.20 Another bit of new numerics introduced in this chapter is the idea of solving two-point boundary value problems by reduction to a matrix linear algebra problem. As a simple illustration of this technique, apply it to the solution of the boundary value problem stated in Problem 5.1.18. Discretized on a grid with uniform spacing Δx, the problem becomes

$$(Y_{j+1} - 2Y_j + Y_{j-1}) - a\Delta x Y_j = F_j \delta x \tag{5.2}$$

using centered differences. This corresponds to a tridiagonal linear system.

Read about tridiagonal solvers in *Numerical Recipes*, and implement a solution of this system using a tridiagonal algorithm. Examine the solution for $F(x) = \sin \pi x$, and look at the convergence as the number of points in the grid is increased. Do the same for the case $F_j = 1$ for $j = m$ and $F_j = 0 otherwise$. In both cases,pay particular attention to what happen as a is made large.

Python tips: A tridiagonal solver is implemented in the chapter script `tridiag.py`

Problem 5.1.21 Write a program which sets up the matrix describing the matrix version of the two-stream equations, given by Eq. **??**. Read about band-diagonal solvers in *Numerical Recipes*. Implement the `bandec` algorithm, or learn to use a band-diagonal solver in the programming environment of your choice. Use this algorithm to solve the two-stream equations for the simple case in which ω_o is constant, the atmosphere is isothermal, and there is no incident beam. Compare the numerical solution with the analytical solution to this problem, paying particular attention to what happens in the optically thick limit. Then try a case where $T(\tau)$ varies with height, and check that the fluxes have the correct behavior in the optically thick limit.

Problem 5.1.22 Explain the joke in the title of Section **??**. Is it funny? If not, why not.

Problem 5.1.23 Evaluate the sum **??** for a single band using a Malkmus transmission function. Compare the OLR to what it would be if there were no reflecting surfaces. Attempt to derive a similar series for the case in which there is a third partially reflecting surface at the midpoint of the atmosphere.

Problem 5.1.24 *Polar CO_2 clouds on Snowball Earth*

This problem requires an implementation of the numerical two-stream solution incorporating band-averaging via exponential sums. If you already have an implementation of the basic numerical solution for a single wavenumber and absorption coefficient, the extension to the exponential sum case is quite straightforward. The exponential sum tables you need are in the dataset subdirectory `Chapter4Data/ExpSumTables`.

Consider a Snowball Earth case with an atmosphere consisting of $100 mb$ of CO_2 mixed with dry air. In the polar winter regions, climate model simulations tend to show that convection shuts off and transient radiative cooling effects lead to an atmosphere that is approximately isothermal. Suppose that the atmosphere is isothermal with temperature T_a, and that the surface has temperature T_g. It is possible for the air to become so cold that CO_2 condensation occurs. To explore the effects of this, suppose that there is a CO_2 ice cloud made of spherical particles with a 5 μm radius, with uniform mass concentration q in the lowest 100 mb of the atmosphere. Using the exponential sum two-stream radiation calculation, compute the effect of this cloud on the spectrum of OLR, for various value of q. Discuss the effect of the cloud on the net OLR. You may pick any reasonable values of T_a and T_g that will help you bring out the essential features of the problem. If you are energetic, you can use the actual Mie theory scattering results to determine the wavelength-dependent scattering properties of the cloud, but to get at the basic behavior, just using a constant scattering cross section estimated crudely from the Mie results, and ignoring cloud absorption, should be sufficient.

Python tips: The basic exponential sum calculation with scattering is implemented in the Chapter Script `TwoStreamScatterEsum.py`, which can be modified to solve this problem.

5.1.4 Absorption of stellar radiation

Problem 5.1.25 *Albedo of dense CO_2 atmospheres*

Venus-like atmospheres should be common throughout the Universe, for planets which have lost their water but still have accumulated enough carbonates to allow the creation of a thick CO_2 atmosphere. Such planets may or may not have a sulfuric acid cloud deck like that of Venus, and so it is important to understand what the albedo would be without clouds. This problem was considered in the text, but without incorporating the joint effects of gaseous absorption and Rayleigh scattering. Now you will bring together both effects.

Consider a planet with a pure CO_2 atmosphere, which has zero surface albedo throughout the spectrum. The surface pressure is p_s, and the surface gravity is 10 m/s^2. CO_2 has significant absorption for wavenumbers below about 15000 cm^{-1}, and in this part of the spectrum you need to use exponential sums to compute the albedo. However, it is not necessary to numerically solve the two-stream equations for this problem, since you can use the analytical solution for the albedo given by the generalization of Eq. **??** done in Problem 5.1.12. You use the exponential sum coefficients to form a weighted sum of calculations for individual absorption coefficents within each band; then you sum results over bands, as usual. You should incorporate pressure broadening and the CO_2 continuum in your calculation of the absorption coefficient, but you may ignore temperature scaling, so as to keep things simple.

The tables you need are found in the Workbook datasets directory `Chapter4Data/ExpSumTables`.

For wavenumbers above 15000 cm^{-1} you can consider CO_2 to be non-absorbing. In this part of the spectrum, you can compute the contribution to the albedo using the analytical Rayleigh scattering solution, but the albedo is wavenumber-dependent and you will need to numerically integrate it weighted by the spectrum of stellar insolation.

Using this procedure, determine the albedo of the planet for both a G star and M star case, for $p_s = 1$, 10 and 100 bar. (Note that the linear pressure scaling of the absorption coefficients may seriously underestimate the absorption in the higher pressure cases). How much does Rayleigh scattering affect the near-IR contribution to the albedo?

Python tips: You can do this problem by modifying the script `TwoStreamScatterEsum.py`, replacing the numerical solution of the two-stream equations by the analytical albedo calculation. You will also need add in code to do the Planck-weighted sum over wavenumbers.

Problem 5.1.26 The dataset subdirectory `UVabsorption` contains ultraviolet absorption cross sections for ozone, SO_2 and SO_2 as a function of wavelength. The cross sections are given in units of cm^2 per molecule. Read these in, convert to m^2/kg and plot them. Determine and plot the ultraviolet absorption optical depth as a function of wavelength for the model ozone profile given in the text. Do the same for a 2 bar pure CO_2 atmosphere on a planet with surface gravity of 10 m/s^2. Do the same for a planet with a 2 bar N_2 atmosphere mixed with 10 $ppmv$ of SO_2, and with 100 $ppmv$ of SO_2.

Problem 5.1.27 *Grey radiative-convective Equilibrium with solar absorption*

In Section **??** we computed the pure radiative equilibrium for a grey gas, including the effects of atmospheric solar absorption. Using techniques similar to those introduced in Problem 4.1.28, extend these solutions to the radiative-convective case, in which solar absorption at the surface can created a troposphere which is adjusted to a dry adiabat. Time-stepping is not required for the solution of this problem, since you can iterate on the tropopause height until equilibrium is achieved. However, since the absorbed solar radiation (which must be balanced by OLR) affects the temperature profile, you can no longer do the problem by fixing the ground temperature at the desired value and computing the temperature of the rest of the atmosphere. One approach is to guess the ground temperature, compute the temperature structure and the OLR, and then vary the ground temperature until top-of-atmosphere balance is achieved. A better approach is to start from the radiative equilibrium solution in the stratosphere which satisfies top-of atmosphere balance, guess a tropopause height, impose temperature continuity there, and then compute the radiative fluxes in the troposphere. You then complete the calculation by adjusting the tropopause height until the upward infrared flux is continuous across the tropopause. This is equivalent to the condition that the flux into the troposphere from above be equal to the upward infrared exiting the troposphere.

Explore the behavior of this system. When does solar absorption substantially cool the troposphere? When does it eliminate the troposphere entirely? When does increasing the infrared absorption coefficient simultaneously warm the stratosphere but cool the troposphere?

Problem 5.1.28 Determine the radiative-convective equilibrium for an atmosphere consisting of 1000$ppmv$ of CH_4 in 1 bar of N_2 subject to a surface gravity of 10 m/s^2. The surface is perfectly absorbing throughout the spectrum. You may ignore Rayleigh scattering. Compare results with and without atmospheric stellar absorption, for both G and M star cases, with stellar constant 1400W/m^2 at the orbit of the planet. To what extent does stellar absorption by CH_4 inhibit its effectiveness as a greenhouse gas?

To do this problem, you will need to use an implementation the homebrew exponential sums radiation code. The necessary exponential sum table for CH_4 is found in the Workbook dataset directory `Chapter4Data/ExpSumTables`. The problem is solved by time-stepping to equilibrium, with adjustment of ground temperature as described in the text.

Python tips: Modify `RadConvEqTrop.py`, which time-steps to radiative-convective equilibrium for a fixed incident solar flux, and has an option to incorporate atmospheric solar absorption.

Chapter 6

Surface energy budget problems

6.1 Workbook

6.1.1 Basic surface balance calculations

Problem 6.1.1 Using Eq. **??**, show that for an optically thin grey gas the emission and absorption coefficients e_a,$e_{a,top}$ and a_+ defined in Section **??** are given by $a_+ = \tau_\infty$ and $e_a = e_{a,top} = \tau_\infty(\bar{T}/T_{sa})^4$, where $\bar{T}$ is the mean atmospheric temperature given by Eq **??**. Compute $(\bar{T}/T_{sa})^4$ and $a_+/(e_a+e_{a,top})$ for the following three cases:

- An all-tropophere ideal gas atmosphere on the dry adiabat, without pressure broadening.
- The same with linear pressure broadening
- The same with linear pressure broadening, but with the temperature patched to in isothermal stratosphere at high altitudes

Under what circumstances is the radiatively-driven temperature jump at the ground unstable to convection?

Problem 6.1.2 *Surface Global Warming fallacy, Part 1*

Suppose we hold the atmospheric temperature fixed on the moist adiabat having a low level air temperature of $280K$ as we increase CO_2 from $300ppmv$ to $600ppmv$. Assuming a moist atmosphere, interpolation from the data in Fig. **??** yields $e^* = .314$ for the original CO_2 and $e^* = .307$ for doubled CO_2. Using these values in a surface energy balance model, determine how much the surface would warm if we ignore turbulent fluxes and assume that the surface balance is purely radiative. Then, assuming the surface to be dry so that there is no evaporation, re-do the calculation including sensible heat fluxes. Now assume the surface is wet and include evaporation as well. For the turbulent fluxes, assume that the surface wind speed is 5 m/s and that the boundary layer relative humidity is 80%. Keep C_D fixed at .0015, and assume that the surface absorbed solar radiation is 200 W/m^2.

Next, for the case including all turbulent fluxes, compute how much the surface warms if you increase the air temperature by $2K$ at the same time CO_2 is doubled. How much does the answer change if you leave the CO_2 fixed at $300ppmv$?

FInally, re-do alll the above calculations assuming the low level air temperature to be $300K$, as is typical of the tropics. For this case, $e^* = .1286$ at $300ppmv$ and $e^* = .1270$ at $600ppmv$

Summarize all your results in a table. What do you conclude about the importance of increase in atmospheric temperature vs. the direct effect of CO_2 on downwelling radiation into the surface? Discuss the role of evaporation in determining how much the surface temperature changes.

Problem 6.1.3 *Making ice in the desert*

You are out in the desert at night, and you would like to make some ice. There is plenty of dry air around. You fill a trough with water. The trough is open at the top so evaporation can occur, but it is well insulated so that no heat is exchanged through the walls. Suppose that the relative humidity of the air is h and the temperature of the air is T_a. Find the conditions on h and T_a under which ice can be made, i.e. the conditions under which the surface temperature of the water can be brought down to freezing if the wind blowing over the water is strong enough. If you are free to make the wind as strong as you like, does your answer depend on the surface radiative cooling factor e^*?

Next, fix T_a at $290K$ and pick a value of h that allows the temperature to go below freezing in the strong-wind limit. Compute the value of ρU needed to make ice for $e^* = .3, .5$ and $.7$. (In reality, e^* would be determined by h with drier values lielding higher e^*). First do the calculation with fixed $C_D = .0015$, and then do it incorporating Monin-Obukhov stable surface layer physics.

The surface of Neptune's moon Triton and the former planet Pluto are believed to be composed primarily of N_2 ice. The atmospheres of these bodies are so thin that their effect on infrared radiation can be neglected. Therefore, the temperature of the surface can be determined from the surface energy balance. At the subsolar point, the surface will heat up until the absorbed solar radiation equals the energy carried away by blackbody radiation (simply σT_g^4 in this case) plus the latent heat of sublimation of the N_2 ice. The saturation vapor pressure of N_2 over the relatively warm subsolar point is much greater than the pressure over the colder parts of the body, and this drives a mass flow away from the subsolar point, carrying latent heat with it. You may assume that the albedo is 60% .

What is the surface pressure of N_2 at the subsolar point if the latent heat flux is negligible? (*Hint:* The gas must be in equilibrium with the solid surface with which it is in contact.) Now suppose that you are given the flux of mass (per unit area) out of the subsolar point. Compute the temperature and pressure as a function of this mass flux. Is there a maximum possible mass flux? Speculate on what processes might determine the mass flux. Can you use the latent heat flux formula derived in the text for this class of problems?

Some relevant data: The solar constant at Triton's orbit is 1.5 W/m^2; at Pluto's it is .89 W/m^2 on average. The latent heat of sublimation of N_2 ice is $2.44 \cdot 10^5 J/kg$.

Problem 6.1.4 Consider a Titanlike situation in which the dominant balance at the surface is between latent and sensible heat fluxes, and all other terms are negligible. Allow for stable boundary layer effects, so that C_D is a function of the Richardson number. Show that if C_D is nonzero, the solution becomes independent of $C_D U$, and derive an expression for the strength of the surface inversion. Show that there is another solution, with $Ri = Ri_c$ and $C_D = 0$, and derive an expression for the strength of the inversion in this case.

Put in numbers corresponding to a CH_4/N_2 atmosphere on Titan, assuming $h_{sa} = .7$, $U = 5m/s$, $z_1 = 10m$ and $z_* = .001m$, for temperatures in the vicinity of $95K$. Do the same for an Earthlike water/air atmosphere with temperatures in the vicinity of $350K$. How valid is the neglect of infrared radiation in the Earthlike case?

Problem 6.1.5 Consider the surface energy budget in a cold climate, in which evaporation can be neglected and the budget consists only of solar absorption, infrared, and sensible heat flux terms. Include the Monin-Obukhov theory for stable surface layers, and set the parameters to Earthlike values.

Plot the surface energy budget as a function of T_g for fixed T_{sa} so as to determine the equilibrium surface temperature. Try this for various values of the absorbed solar radiation $(1 - \alpha_g)S_g$, and discuss the strength of the inversion and the effect of turbulence suppression. Are there multiple equilibria? Are there sharp transitions between strong-inversion and weak-inversion cases as the solar absorption is changed? Repeat this calculation for some much colder and much warmer values of T_{sa}, so as to alter the importance of the infrared radiation term.

To keep things simple, you may assume that e_a is held constant at a value of 0.2 throughout this problem, and that the ground has unit emissivity. *Hint:* You should first get a feel for the problem by making a set of energy balance graphs analogous to Fig. **??**, but at some point you may wish to write a simple Newton's method routine to solve the surface budget equations iteratively. You can use the iteration to implement a function `Tg(Tsa)` using your favorite programming language.

6.1.2 Evaporation and sublimation

Problem 6.1.6 Re-do the graph in Fig. **??** for parameters which make the surface layer more stable: smaller U, smaller relative humdity, and/or smaller solar absorption

Problem 6.1.7 Sublimation steals energy that could be more effectively used for melting. Study this by computing the latent heat flux for the cases in Fig. **??** and determining how much additional melting could be sustained if this energy went into melting instead. Study the issue further by recomputing the figure with lower and higher relative humidity. Increasing the wind speed increases both sensible and latent heat fluxes; does the net effect enhance or retard melting?

Problem 6.1.8 Use the linearized form of the surface energy budget to compute $T_g - T_{sa}$ in the weak evaporation limit, and use the resulting expression to discuss how the latent heat flux increases with temperature. Assume the boundary layer to be governed by neutrally stratified theory, so that C_D can be regarded as independent of temperature. For simplicity, you may also assume $e^* = .8$ throughout this problem, and neglect variations with temperature.

Problem 6.1.9 This problem explores the behavior of the weak evaporation limit when the surface layer becomes stable enough to significantly suppress turbulence. You may assume that the atmosphere is also cold enough that the buoyancy is determined by temperature alone, and is unaffected by the concentration of the condensible vapor. First determine the ground temperature $T_{g,o}$ at which turbulence is completely suppressed. You do this by setting $C_D = 0$ in the surface balance equation and finding an expression for the resulting radiatively-determined T_g. Then use this value to determine the circumstances under which $Ri = Ri_c$ for this temperature. This determines when the assumption $C_D = 0$ is consistent. Finally, assuming this condition to be satisfied for some given air temperature $T_{sa,o}$, discuss how T_g varies as T_{sa} is increased beyond $T_{sa,o}$ (holding other parameters fixed), retaining terms out to second order in $T_{sa} - T_{sa,o}$. Use the result to say how the evaporation behaves as a function of temperature in the weak evaporation limit. Illustrate your result by putting in some numbers characteristic of a cold Earthlike climate with weak absorbed solar radiation, with an atmosphere consisting of air and water vapor. For simplicity, you may assume $e^* = .8$ throughout this problem, and neglect variations with temperature.

Problem 6.1.10 As discussed in the text, the limiting strength of the inversion in the strong-evaporation limit is given by the temperature T_o where $F_L(T_{sa}, T_o) = 0$, i.e. where the latent heat flux vanishes. Using the expression for F_L, show that this occurs when $p_{sat}(T_o) = h_{sa} p_{sat}(T_{sa})$, where h_{sa} is the relative humidity of the air at the upper edge of the surface layer. Use the simplified exponential form of Clausius-Clapeyron to derive an expression for $T_{sa} - T_o$. How rapidly does this increase with T_{sa}? Use the results to put an upper bound on the growth of evaporation with temperature in the strong evaporation limit. Put in some numbers corresponding to Titan, and to a very hot Earth.

6.1.3 Ice

Problem 6.1.11 Determine equilibrium ice thickness for a slab of ice which allows solar penetration into the ice as discussed in text, but with the following extensions: (1) Assume that a geothermal heat flux F_i is applied at the base of the ice, and (2) Allow for the possibility that the ice is so thin or transparent that some solar radiation penetrates the ice and escapes into the ocean below.

First do the calculation assuming that solar radiation that leaves the bottom of the ice is carried away in the ocean and does not need to re-escape through the ice. Then say what happens if the energy is absorbed locally and heats the water below the ice, leading to the heat being delivered to the base.

For the purposes of this problem, you may neglect the effects of sublimation and accumulation at the top of the ice.

Note that aside from its possible application to tropical ice on Snowball Earth, this "thin, clear ice" calculation applies to ice-covered Antarctic Dry Valley lakes, which some see as a modern analog for Snowball Earth tropical conditions.

Problem 6.1.12 Consider sea ice of thickness d on a Snowball Earth, in a region where the surface temperature is in equilibrium with an absorbed solar flux of $50W/m^2$. The ice has unit infrared emissivity. What is the surface temperature ignoring the heat flux through the ice? Then, assuming that the base of the ice has a temperature of $270K$, compute the surface temperature as a function of d, taking into account the heat flux through the ice. You may assume that the temperature profile within the ice is in equilibrium. Ignore the effects of solar flux penetration into the ice.

Problem 6.1.13 In the text we determined the effect of solar flux penetration into ice assuming the decay rate to be exponential. In reality, the decay rate is weaker than exponential, because of wavelength dependence of the absorption coefficient. Re-do the calculation assuming that the solar flux decays according to a power law of the form $1/(1 + a \cdot z^b)$, where z is distance from the surface and $b > 0$ is a decay exponent. You may assume the ice to be infinitely thick. What is the deep ice temperature?

Problem 6.1.14 The current permanent glaciers on Mars are made of water ice, but suppose that at some past time Mars had CO_2 ice glaciers. Let the thickness of the glacier be h at some point. In order for melting at the base to be possible, the basal presure has to be above the triple point pressure. What is the critical height? Suppose that the surface temperature of the glacier is $200K$, which is approximately in equilibrium with a 1.5 bar CO_2 atmosphere. Assuming the height to be above the critical height, compute, as a function of height, the magnitude of geothermal heat flux needed to melt the base. At $200K$ the thermal conductivity of CO_2 is is about 0.5 $W/m \cdot K$.

Basal melting is important, because it lubricates the glacier and allows it to surge into regions that are warm enough to ablate it rapidly.

6.1.4 Radiative-convective equilibrium coupled to a surface budget

Problem 6.1.15 In Problem 5.1.27 you computed the radiative-convective equilibrium for a grey gas atmosphere assuming that the surface coupling was so strong that the ground temperature T_g was always equal to the overlying air temperature T_{sa}. Using the same basic solution approach, generalize the problem to the case in which T_g differs from T_{sa}, and the relation between the two temperatures is determined by the surface energy balance. To keep things simple, you may assume that the surface balance involves only radiative and sensible heat fluxes. Further, you may assume that the atmosphere is transparent to solar radiation. Note that even though the atmosphere absorbs no solar radiation, you can't do this problem by fixing the surface temperature and computing the *OLR* afterwards, since the surface budget depends explicitly on the surface absorbed solar radiation $(1-\alpha_g)S_g$. You will need to use the solution technique outlined in Problem 5.1.27.

Note that when doing convective adjustment in a problem like this, the convective adjustment should be based on the low level air temperature T_{sa}, not the ground temperature. The ground cannot convect upward! Heat must first be transferred to the air by turbulent and radiative fluxes. As long as the troposphere doesn't disappear completely, one doesn't need to do anything to explicitly take this heat transfer into account. It is implicitly mixed throughout the troposphere by convection, and the transfer of solar heating of the ground into the troposphere is implicitly taken care of when one imposes the condition of net (solar plus *IR*) radiation balance at the tropopause.

Assume $R/c_p = \frac{2}{7}$, and that the grey absorption coefficient increases linearly with pressure. Fix the absorbed solar radiation at 250 W/m^2 and explore the behavior of the system as a function of $\rho C_D U$ and τ_∞. At large $\rho C_D U$ the solution should reduce to the previous one, with $T_g \approx T_{sa}$. How does the surface budget affect the tropopause height? How much does it affect T_{sa}? How much does it affect the stratospheric temperature? Make sure to cover both optically thick and optically thin cases. Discuss the magnitude of the surface temperature jump in the optically thin (Marslike) case. Finally, look at the warming in T_g and T_{sa} you get when you increase τ_∞ from 1 to 1.1 . How important is the surface budget in determining the amount of low-level warming?

Problem 6.1.16 Extend the inquiry in Problem 6.1.15 to the real-gas case. Specifically, use the homebrew exponential sums radiation code to compute radiative-convective equilibrium for CO_2 mixed with dry air on the dry air adiabat. You may ignore temperature scaling of the absorption coefficients and the effects of atmospheric solar absorption. Include only radiative and sensible heat terms in the surface budget, and assume a constant C_D. You solve the problem by time-stepping to equilibrium in a fashion almost identical to that described in Section **??**. The only exception is that instead of adjusting T_g at each time step so as to move towards a state of top-of-atmosphere balance, you adjust the air temperature T_{sa} and then determine the corresponding T_g by solving the surface energy balance equation using the downwelling infrared from the atmosphere as one of the inputs.

Carry out a calculation of this type, and explore how the surface temperature, air temperature, stratospheric temperature and tropopause height depend on the surface wind speed. Try some very low CO_2 values where the atmosphere is very optically thin, and try partial pressures up to 100mb, where the atmosphere is more optically thick. You can keep the surface air pressure fixed at 1000mb for these calculations, and assume Earth gravity. Remember that convective adjustment should be done based on T_{sa}, not on T_g.

If you can run the `ccm` radiation code, you can alternatively do this problem by timestepping a `ccm`-based radiative-convective model to equilibrium. In that case, you can include the joint effects of CO_2 and water vapor, determing the latter assuming a fixed relative humidity. In that case, it also makes sense to include evaporation in the surface budget.

Python tips: The radiative-convective calculation for the homebrew model is done in the Chapter Script `RadConvEqTrop.py`, and that for `ccm` is done in `ccmRadConv.py` . Both are included in the Chapter Scripts directory for Chapter **??**. These scripts do not include a surface balance, but need only slight modifications in order to allow T_g to be computed by solving the surface balance at each time step.

Problem 6.1.17 *Surface Global warming fallacy, Part 2*

In this problem you will extend the inquiry of Problem 6.1.2 by simultaneously determining the change in ground temperature and air temperature when CO_2 is increased. This is done by simultaneously satisfying the top-of-atmosphere and surface balance conditions. We'll do this based on real-gas radiation calculations from the `ccm` radiation model incorporating both CO_2 and water vapor, but to keep things simple we'll employ an all-troposphere model so that the calculations can be done using polynomial fits to the radiation computations. With this approach, you don't need to be able to run the `ccm` model yourself, but more importantly the simplification makes it much easier to understand what is going on.

In Table **??** we gave polynomial fits to $OLR(T_g, CO_2)$ for fixed relative humidity, assuming that $T_g = T_{sa}$. With the surface budget incorporated, we must allow the two temperatures to differ, and because the atmosphere is not completely optically thick, this affects the OLR. We can write the OLR using a variant of Eq. **??** as follows

$$OLR(T_{sa}, T_g) = OLR(T_{sa}, T_{sa}) + (1 - a_+(T_{sa}))(\sigma T_g^4 - \sigma T_{sa}^4) \tag{6.1}$$

where a_+ is the atmospheric absorption factor discussed in the text. The first contribution to OLR, i.e. $OLR(T_{sa}, T_{sa})$, is obtained using the polynomial fit in Table **??**. This function, as well as a_+, depends on CO_2 in addition to temperature. To keep things simple, assume the atmosphere to be transparent and nonscattering, so that the OLR is in equilibrium with an absorbed solar flux $(1 - \alpha_g)S$. This balance condition gives you one relation between T_g and T_{sa}

The surface balance condition imposes an additional relation between T_g and T_{sa}, which closes the problem. This is the same surface balance condition as discussed in the text, and which you used in Problem 6.1.2. It requires that you know the surface cooling factor e^*, and the way it depends on temperature. In the surface budget, you may assume that C_D is fixed at 0.0015 . Include both sensible and latent heat fluxes in your surface budget.

To complete the calculation, you need to be able to determine how $(1 - a_+)$ and e^* vary as a function of T_{sa} and CO_2. Tabulations giving the variation are provided in the datasets `ccmTransWet.txt` and `SurfEstarWet.txt` in the Workbood datasets directory for this chapter. These were computed using the `ccm` radiation model with 50% relative humidity aloft. Values for arbitrary temperature and CO_2 within the range of the table can be found using interpolation.

The solution procedure is simple. You guess T_{sa}, and then solve for T_g using the surface energy balance. Solving for T_g requires a small Newton's method iteration if you use the full form of the surface budget, but if you are in a regime where the surface budget can be linearized then you can actually do this step analytically, as discussed in the text. Either way, once you have T_g, you can compute the OLR. You then iterate on T_{sa} until the top-of-atmosphere budget is satisfied.

Note that when $(1 - a_+(T_{sa}))$ is small or when the surface coupling is so strong that $T_g \approx T_{sa}$, then the OLR expression allows T_{sa} to be determined nearly independently from the surface budget. Except for optically thin atmospheres with weak coupling, the surface budget usually imposes only a small correction on the T_{sa} you get by ignoring the second term in the OLR expression above.

Write a program which implements this procedure. First fix CO_2 at $300ppmv$, and find values of $(1 - \alpha_g)S$ that give you values of T_{sa} near $260K$, $280K$ and $300K$ with evaporation included and a wind $U = 5m/s$. Then double CO_2 and recompute the temperatures. How much of the increase in T_g is due to the increase in the air temperature? How much of the increase is due to the direct effect of CO_2 on downwelling radiation for fixed air temperature? Compare the results with the temperature increase you get by fixing $T_g = T_{sa}$.

Then, re-do the series of calculations for a case with significantly weaker winds and a case with significantly stronger winds. Finally, with $U = 5 m/s$, how do your results change if you eliminate the latent heat fluxes, as would be the case over a dry surface?

If you like you can also do this problem by time-stepping the full `ccm` calculation to radiative-convective equilibrium. See Problem 6.1.16 for hints on how to do that.

Python tips: Remember that the `interp` class in `ClimateUtilities` makes polynomial interpolation easy. To interpolate to an arbitrary temperature and CO_2 from the table, first do an interpolation to the temperature you want for all the CO_2 columns, and then do an interpolation in $\log(CO_2)$. The transparency and surface cooling coefficients are computed in the Chapter Script `ccmSurfrad.py`, but if you recompute the coefficients with different gravity, surface pressure or humidity you will need to make sure to use an OLR polynomial which is computed consistently.

Chapter 7

Seasonal Cycle problems

7.1 Workbook

Python tips: The Chapter Scripts `solar.py`, `plot_solar.py` and `FluxExplorer.py` will be useful in doing the problem in this Workbook section. The script `solar.py` includes routines to carry out the integration needed to determine the season angle as a function of time, and can be imported into other scripts that need insolation calculations. The function `getOrbit(...)` defined in `FluxExplorer.py` retrieves the orbital parameters for Earth from a data file given the time expressed in thousands of years relative to the present.

Some of the problems call for use of a graphics tool which has not been previously introduced, namely contour plotting. In the Python courseware, the contour plots can be done using the function `contour(...)` in the `ClimateUtilities` module.

7.1.1 Distribution of insolation

Problem 7.1.1 Using Eq. **??**, estimate the duration of daylight at the latitude where you are living, at the time of year at which you are solving this problem. Use the current obliquity of Earth, which is 23.45^o. Compare your results with observations, preferably taken by yourself. Using Eq. **??**, estimate how much solar power could be generated by a one square meter 10% efficient solar panel pointed straight up at the sky at this time, assuming that 80% of the solar energy incident on the atmosphere makes it through to the ground. Thirty thousand years ago, the obliquity was 22.25^o. How would your results have been different at that time?

You may make use of the approximation that the length of the day is short compared to the length of the year, and approximate the orbit as being circular.

Problem 7.1.2 Determine the latitudinal pattern of diurnally averaged insolation for a planet with small but nonzero obliquity, assuming that the planet's length of day is short compared to its year. Assuming a circular orbit, average this over the course of a year to obtain the annual mean insolation pattern. Compare the annual mean results to the more precise numerically determined flux factor for obliquities of 10^o, 20^o and 30^o. Make sure to retain enough terms in your approximation to see the lowest order deviation from the zero obliquity case.

Problem 7.1.3 Estimate what the summer temperature of Antarctica would be if you took away the ice sheet, leaving bare rock. Use the polynomial OLR fit in Table **??** to determine the infrared cooling, and neglect horizontal heat redistribution by dynamical heat transports.

Problem 7.1.4 The Earth's obliquity fluctuates between 22^o and 24.5^o on a time scale of roughly 40,000 years. Assuming that the albedo is 20%, compute the latitudinal profile of the *difference* in absorbed solar radiation between these two states. Convert these fluxes into temperature change using a climate sensitivity of $2W/m^2K$. This gives you an estimate of the temperature change over extensive oceans, including the effect of water vapor feedback but neglecting the additional sensitivity due to possible melting of sea ice or land glaciers.

Problem 7.1.5 *High obliquity waterworlds*

This problem examines the climate of high-obliquity waterworlds. These are of interest because a planet without a large moon such as Earth's would generally pass through high obliquity episodes at various times. Would such planets be habitable?

Consider a waterworld planet in a circular orbit, whose thermal response time is sufficiently long that the seasonal cycle is weak and the surface temperature responds mainly to the annual average insolation. The albedo is 0.2, and the local solar constant is $L_\circledast$. The planet has an Earthlike atmosphere, whose OLR can be approximated using the polynomial fit in Table **??**. For a planet in an Earthlike orbit, with $L_\circledast = 1370 W/m^2$, plot the surface temperature as a function of latitude for obliquities of 45^o, 50^o, 55^o, and 60^o, assuming a CO_2 value of 1000 *ppmv*. Then, for a fixed 60^o obliquity, examine the range of values of CO_2 and $L_\circledast$ which yield a globally ice-free climate. Determine the threshold $L_\circledast$ below which the planet would definitely be in a Snowball state (as indicated by below-freezing temperatures everywhere), with CO_2 at 10^5*ppmv*.

For the purposes of this problem, you may neglect horizontal heat transports and assume the planet to be in equilibrium with the absorbed solar radiation at each latitude.

Problem 7.1.6 The obliquity of Mars undergoes more extreme variations than that of Earth. Consider a time when Mars has an obliquity of 55^o. Assume that the planet has no oceans, and that the polar glaciers have dissipated so that the albedo is 20% at the poles. What is the temperature of the summer pole, assuming that there is no greenhouse effect and that the polar temperature is in equilibrium with the local solar absorption (i.e. that there is no horizontal heat transport)? If you put a water ice-cap with 60% albedo there, would it melt or stay below freezing? How cold would it be? (In reality, even an ice cap that is too cold to melt may sublimate and be redeposited elsewhere on the planet). For the glaciated case, use the OLR data in Fig. **??** to determine the surface pressure of a pure CO_2 atmosphere that would be needed in order to bring the glacier to the melting point.

For this problem, you may assume Mars to have a circular orbit, though in reality the eccentric orbit of Mars will episodically render one of the summer poles much warmer than the estimates obtained here.

Problem 7.1.7 Consider a planet with a solid surface, having an atmosphere which is thick enough to average out the diurnal cycle but which has a response time short compared to the length of the planet's year. In this case, the temperature at any given latitude is in equilibrium with the absorbed solar flux computed from Eq. **??**, provided that the lateral heat transport in the atmosphere is neglected. This calculation also yields a reasonable approximation to what the temperature would be in the interior of large continents, in the absence of lateral heat transports.

Using the OLR polynomial fit for fixed relative humidity in Table **??**, determine the temperature profile of this planet at the solstice and equinox, assuming Earth's present solar constant and an obliquity of 23^o. Do the calculation for a rocky planet with albedo 0.2, and a Snowball state with albedo 0.6, and a snow-covered Snowball state with albedo 0.8. Examine how the results change as you increase the CO_2 from 100 *ppmv* to 10^5*ppmv* (100 *mb*). Then, fix the CO_2 at 1000*ppmv* and look at how the temperature pattern changes as you vary the obliquity from 0^o to 90^o.

You should cut off the calculation where the temperature falls below $200K$, or goes above $350K$, since that is beyond the range of validity of the polynomial fit. In reality, the hot cases would go over into a runaway state, if the relative humidity continued to be constant and if the unrealistic neglect of lateral heat transport really were valid.

Problem 7.1.8 *Triton*

Neptune's moon Triton has probably the most unusual seasonal cycle in the Solar System. Like most moons, it is tide-locked to its planet, in Triton's case with a period of $5.67d$ (in the retrograde direction, though that does not matter for the purposes of this problem). Uniquely among moons, however, its orbit is in a plane which is quite steeply inclined to Neptune's equator. The orbital plane precesses about the normal to Neptune's orbital plane with a period estimated to be about 637 Earth years, at the same time that Neptune orbits the Sun with a period of 165 Earth years. Putting the two effects together, the latitude of the subsolar point follows the empirical formula

$$\sin\delta = .4636\cos(a_o - a_1 t) + .3495\sin(a_o - b_o + (b_1 - a_1)t) + .0251\sin(a_o + b_o - (a_1 + b_1)t) \tag{7.1}$$

where $a_o = 1.332$, $a_1 = .006650$, $b_o = .3229$, $b_1 = .0001721$. In this formula, angles are given in radians and t is the time given in Earth years AD. Thus, the formula can be used to determine the latitude of the Sun on Triton occurring at an actual date. Because Triton is an icy body rich in N_2 and CH_4 ices that can warm up sufficiently to give off significant amounts of gas when illuminated, the complex motion of the subsolar point leads to a rich variety of atmospheric and surface evolution features on Triton.

The solar constant at Neptune's orbit is $1.51 W/m^2$. Using Eq. **??**, plot the time series of diurnally averaged flux on Triton at a both poles, at the equator, and at latitudes of $\pm 45^o$. Plot the time series of surface temperatures that would be in equilibrium with these fluxes, and use the Clausius Clapeyron relation to plot the saturation vapor pressure of N_2 and CH_4 that would be in equilibrium with these fluxes.

Because heat takes time to diffuse through a solid, to a first approximation the subsurface temperature can be determined by averaging the surface temperatures obtained above over longer periods; the longer the averaging period, the deeper the layer the temperature corresponds to, with averaging times of an Earth year yielding the ice temperature at around a meter of depth, increasing to around $30m$ for an averaging period of 100 Earth years and around $100m$ for 1000 Earth years. Plot time series of the averaged temperature at selected latitudes, using averaging periods of one, ten, a hundred and a thousand Earth years. Also show the N_2 and CH_4 saturation vapor pressures at these periods. The pressures are of interest because when sufficiently high pressures build up it can fracture ice and lead to cryovolcanism. Note that the subsurface temperatures you have estimated are underestimates of the actual subsurface temperature, since the penetration of solar radiation into ice can lead to considerably elevated temperatures, as discussed in Section**??**.

Problem 7.1.9 *Path of the Subsolar Point*

Show that for a nonrotating planet with a latitude-longitude coordinate system based on an axis with obliquity γ, the longitude λ_o of the subsolar point is given by

$$\tan\kappa = (\cos\gamma)(\tan\lambda_o) \tag{7.2}$$

where κ is the season angle. When solving this formula for λ_o, one must take care to choose the branch of $\tan^{-1}$ which yields an angle which increases smoothly from 0 to 2π radians in the course of the orbit. *Hint:* Project the equatorial circle, with all its longitude labels, onto the plane of the orbit. The result will be an ellipse, labeled by longitude. This ellipse rotates along with the season angle, and you now only need to find the longitude of the point where the ellipse intersects a line segment extending from its center to the center of the sun.

Discuss the limit of zero obliquity, as a check that the equation is behaving sensibly. What should the behavior be when the obliquity is close to 90^o? Compare your expectation with the behavior of the above equation for obliquity close to 90^o.

If the planet is rotating about the axis with constant angular speed Ω, then the longitude of the subsolar point becomes $\lambda_{\circledast} = \lambda_o(t) - \Omega t$.

The *solar day* is defined as the time elapsed between successive local noons – i.e. the time between successive crossings of the local longitude by the subsolar point. Note that this is independent of latitude. Find a formula for $\lambda_{\circledast}(t)$ for the case $\gamma = 0.$. Show that if the year contains n stellar days, then it contains $n - 1$ solar days.

Find an approximate expression for the duration of the solar day assuming the duration of the stellar day to be short compared to the orbital period. Use this formula to determine how Earth's solar day would vary over the year if its orbit were circular with a period of $365.25d$, given a present stellar day of $86164s$. Then, numerically determine the sequence of solar day lengths for a planet in a circular orbit with period $400d$ and a stellar day of $100d$. Show results for obliquities of 0^o, 20^o and 45^o.

Using the formula for $\lambda_{\circledast}$ together with the formula for the latitude of the subsolar point ($\delta(t)$) given in the text, plot the trajectory of the subsolar point in the longitude-latitude plane for the following cases, assuming the planet to be in a circular orbit: (a) A planet with an orbital period of $400d$ and a stellar day of $40d$, and (b) A planet with an orbital period of $400d$ and a stellar day of $100d$. Show results for obliquities of 20^o, 45^o and 80^o.

Problem 7.1.10 Consider a planet in a circular orbit, with an orbital period of $400d$ and a stellar day of $100d$, with 45^o obliquity. Use the results of Problem together with the expression for the zenith angle in Eq **??** to make a contour plot of the diurnally averaged flux factor in the longitude-latitude plane, for each of the three solar days in the planet's year. Then show the annual-average flux factor. Should the annual mean depend on longitude? How do these results compare with what you would get by ignoring within-day variations in the latitude and longitude of the subsolar point, and simply averaging over the hour angle?

Hint: You evaluate the time-dependence of $\cos\zeta$ by computing $\delta(t)$ and $\lambda_{\circledast}(t)$ and plugging the values into Eq. **??**, writing the hour angle as $h = \lambda - \lambda_o dot(t)$, where λ is the longitude. The time average at any given latitude and longitude can no longer be done analytically, and must be done instead by numerical quadrature. Remember to zero out the flux for times when the sun is below the horizon.

Problem 7.1.11 *A walk in the sun*

As preparation for doing this problem, first read Geoff Landis's story "A Walk in the Sun"

Astronaut Katie Mulligan has crashed on the Moon, right at the subsolar point. She can be rescued, but it will take 28 days (a Lunar day) for the rescue mission to get there. At the time of the crash, the latitude of the subsolar point is ϕ_o. Her space suit can maintain life support over that time period *but* it is solar powered and has essentially no battery storage capability, so she will need to stay in sunlight for the whole time. How fast does she have to walk if she chooses to walk along her initial latitude circle? Suppose instead she walks along a longitude line directly toward the summer pole and waits there. Under what circumstances is this a better strategy?

Recall that the Moon is tide-locked to the Earth and orbits very nearly in the Earth's equatorial plane. Therefore, it shares Earth's obliquity. Note also that the latitude of the subsolar point will change somewhat in the course of a Lunar day, so you should take this into account, though the effect is slight.

7.1.2 Thermal Inertia

Problem 7.1.12 *Climate sensitivity and equilibration time*

In circumstances where the thermal inertia is determined by the top-of-atmosphere energy budget, the radiative damping coefficient $b = dOLR/dT$ is the same coefficient that determines climate sensitivity. This observation links the ultimate amount of warming due to CO_2 increase with the time it takes to approach this equilibrium. All other things being equal, a more sensitive climate will also have a greater time lag between forcing and response. This link was first pointed out by James Hansen in the 1980's.

Considering only annual-mean conditions, suppose that the Earth is in equilibrium at a temperature T_o, for some given CO_2 concentration. Then, CO_2 is instantaneously doubled, leading to a reduction in OLR of 4 W/m^2 for any given temperature; assume that this leaves the slope b unchanged. Find and plot the subsequent time evolution of temperature for a standard case with $b = 2W/m^2K$ and for a case with stronger amplifying climate feedbacks having $b = 1W/m^2K$. In both cases, assume a mixed layer depth of $50m$.

Problem 7.1.13 Solve the problem

$$\rho_o c_{po} H \frac{dT}{dt} = (1-\alpha)S - \sigma T^4 \tag{7.3}$$

analytically, for arbitrary initial temperature T_o. Analyze the behavior of the system, and compare the analytical results to the approximate results obtained by either linearizing about the equilibrium temperature or holding the infrared cooling rate fixed at σT_o^4. *Hint:* This is essentially an intricate exercise in the use of partial fractions. You need to factor something of the form $1/(A+T^4)$ into a sum of four simpler fractions. You can plot the results analytically if you plot $t(T)$ instead of $T(t)$.

Note that this problem can be solved numerically by ODE integration with only a trivial amount of programmer time, and microseconds or less of computer time. It probably takes less computer time to solve the problem numerically than it does to evaluate the expression yielded by the analytic solution. However, it is good to exercise ones' skills at algebraic manipulation from time to time.

Problem 7.1.14 *Thermal inertia for a shallow troposphere*

When the tropopause is high, then convective mixing allows heat to be stored in essentially the entire mass of the atmosphere. When the convection extends only over a shallow layer, however, less mass is available to store heat and the thermal inertia is correspondingly lower. As a simple example of this consider a dry atmosphere with an isothermal stratosphere. Specifically assume the troposphere is on the dry adiabat from the ground up to the point where the temperature equals a fixed stratospheric temperature T_{strat}, and is isothermal at higher altitudes. Compute the thermal inertia coefficient $\mu = dE(T)/dT$ for this atmosphere, and discuss how it varies with T.

Problem 7.1.15 *Latent heat and thermal inertia*

In the text we determined the thermal inertia of a dry atmospheric column using the dry static energy. If the atmosphere instead is in equilibrium with a condensible reservoir such as a water ocean, then it takes more energy to increase the surface temperature by $1K$ because one has to take into account the increase in latent heat storage of the atmosphere. This problem considers this effect in two simple limits.

In the first limit, we consider the condensible substance to be dilute, as is the case for water vapor on the present Earth. In this case we can use the dilute moist static energy formula given in Section **??**. Suppose that the stratosphere has negligible heat storage, and that the moist static energy can be considered constant for the purpose of computing thermal inertia. Suppose that the condensible substance is saturated at all heights. By evaluating the moist static energy at the ground, find an expression for the energy storage as a function of surface temperature. The thermal inertia coefficient μ is the derivative of this with respect to surface temperature. Evaluate this at $260K$, $280K$, $300K$ and $320K$ for saturated water vapor on Earth. Evaluate it for saturated methane in an N_2 atmosphere on Titan, for a suitable range of temperatures. Express your answer in terms of an equivalent liquid water mixed-layer depth. How do the values compare with what you would get in the same situations using dry static energy? Note that when the atmosphere is unsaturated, the moist static energy per unit mass isn't constant, and the determination of the thermal inertia coefficient is correspondingly more complicated.

In the opposite extreme, we take an atmosphere that is on the single-component condensing adiabat, with temperature profile given by the dew-point formula. This is an appropriate model to use in conjunction with the formation or sublimation of a CO_2 ice cap in the polar winter and spring of Present Mars or Early Mars, and it can also apply to a steam atmosphere undergoing a runaway greenhouse. For this kind of atmosphere, the energy storage is just the latent heat times the mass per unit area of the atmosphere, which by Clausius-Clapeyron can be written as a function of surface temperature. Find the thermal inertia coefficient by taking the derivative of heat storage with respect to surface temperature. How does the gravity affect the thermal inertia? Evaluate the thermal inertia coefficient as a function of temperature, and express in terms of equivalent mixed layer depth for a CO_2 atmosphere under Mars gravity. Do the same for a pure water vapor atmosphere under Venus gravity, at temperatures up to $400K$.

Problem 7.1.16 *Atmospheric thermal response time over a deep mixed layer*

For an atmosphere over a very deep mixed layer the surface temperature can be considered essentially fixed at a specified temperature T_o as the atmospheric temperature varies in the course of the seasonal or diurnal cycle. Find an expression for the thermal rexation time appropriate to this case. Aside from using a thermal inertia coefficient appropriate to the atmopheric mass, the only difference with the calculations of relaxation time carried out in the text is in the form of the flux coefficient b, which determines how the atmospheric heating or cooling depends on the atmospheric temperature. In this case the atmospheric heat budget involves not only $OLR(T)$, but also the surface flux terms. The latter are treated using the linearized flux coefficients discussed in Chapter **??** (see esp. Table **??** and the associated discussion) The linearization should be done about an atmospheric temperature chosen such that the atmosphere is in equilibrium. In determining this temperature, allow for the effect of a time-independent absorbed solar flux in the atmosphere.

Discuss the behavior of the relaxation time as T_o and surface wind is varied. Note particularly the implications of the increasing importance of latent heat flux as T_o increases. How does the atmospheric absorbed solar radiation affect the result?

To keep this problem simple you may assume that the thermal inertia coefficient μ has its dry adiabatic deep-troposphere value $c_p p_s/g$. At warmer temperatures where latent heat flux is significant, for consistency one should actually consider the effects of latent heat storage in the atmosphere. When the atmosphere is saturated, this can be done using the results of Problem 7.1.15, but for an unsaturated atmosphere finding a proper formulation is quite challenging, since the moist static energy is no longer uniform and the effect of moisture depends a lot on whether or not it is condensing.

Use your estimate of the relaxation time together with the analytic solution for a mixed-layer seasonal cycle driven by sinusoidally varying forcing to estimate the amplitude of the seasonal cycle of atmospheric temperature at latitude 45^o over a deep mixed layer. Assume that the atmosphere absorbs 20% of the incident solar radiation. Determine the phase lag between the insolation and the temperature.

Problem 7.1.17 *Sea ice growth and decay*

The growth and decay of a sea ice layer provides a form of thermal inertia, since the latent heat of fusion must be taken into account in determining the heat storage in the atmosphere-ocean column. Assume that all freezing takes place at the sea ice base, and that the diffusion equation is nearly in equilibrium within the ice layer so that the temperature profile is linear. Suppose the base has temperature T_f and the ice surface has temperature T_s, both of which are constant. As an idealization, assume that all the heat released released by freezing at the base must be taken away by diffusing through the ice layer, and that this is the limiting factor in ice growth. (In reality much of the heat in the early stages escapes through open-water leads in the ice layer, and it is also possible that ocean currents could take away some of the heat.) Show that the growth of the ice thickness h is governed by the equation

$$\rho_i L_f \frac{dh}{dt} + \Phi_o = \kappa_T \frac{T_s - T_f}{h} \tag{7.4}$$

where κ_T is the thermal conductivity, L_f is the latent heat of fusion, ρ_i is the density of the ice and Φ_o is the heat flux delivered to the ice base by geothermal heating or other means. Generally, T_f can be taken to be the freezing point of sea water.

Find the solution to this equation. With $\Phi_o = 0$, how long does it take for ice to grow to a thickness of $5m$ for $T_f - T_s = 5K$? For $T_f - T_s = 20K$? Next set Φ_o to the present Earth geothermal heat flux, and determine how long it takes to reach the equilibrium ice thickness for a typical annual mean Snowball Earth polar temperature of $200K$, and a typical Snowball tropical temperature of $240K$.

The preceding coniderations determine how much ice is built up during the growth season. The resulting thermal inertia comes into its own during the melt season, which has different physics since ice can melt from the top. Using the surface energy balance models developed for melting ice in Section **??**, determine the time required to melt a layer of sea ice of thickness d, as a function of the surface solar flux and the air temperature. You may assume that the albedo of melting ice is 0.5, and that the wind speed is 5 m/s. Discuss the difference between melting rates with Monin-Obukhov vs. neutral surface layer physics. Put in some numbers corresponding to polar summer conditions, and determine how long it takes to melt a 5 m layer.

7.1.3 The diffusion equation and its uses

Problem 7.1.18 You are spending the winter inside a hemispherical igloo with radius of $3m$. The walls are $0.25m$ thick. The floor is insulated with caribou skins, so that the heat loss throught the floor can be neglected. The leakage of air through the entrance is also negligible. The outside temperature is $220K$. You and your companion put out a total of $300W$ on average (because of all the food you eat), and you burn a seal oil lamp that puts out another $100W$. What is the temperature inside the igloo if it is made of snow? If it is made out of blocks of ice?

Problem 7.1.19 Numerical methods for the diffusion equation

In this chapter, we made use of some new numerical techniques which have not been introduced earlier, namely the methods for numerical solution of the time-dependent diffusion equation in one dimension. Read about methods for the diffusion equation in *Numerical Recipes*. For the calculations we need, the simplest algorithm (second-order centered finite differences, with leapfrog or midpoint-method time stepping) will suffice. Implement the method in the computer language of your choice. First, test your implementation on a basic configuration consisting of an ice layer of thickness $100m$ and uniform diffusivity, resting on a water layer which keeps the lower boundary of the ice at a fixed temperature of $271K$. Impose an insulating boundary condition ($\partial_z T = 0$) at the upper boundary, and examine the approach to an isothermal state assuming the ice temperature is initially $200K$ independent of depth, except for the lower boundary point.

Next, try a few extensions of the basic problem. Re-do the calculation assuming that there is a $5m$ deep layer of low-diffusivity snow on top of the ice, with the insulating boundary condition applied at the snow surface. Describe how the time evolution changes, concentrating on the time required for the ice temperature to reach equilibrium. Finally, replace the insulating upper boundary condition with the assumption that the surface loses heat by radiating at a rate σT_s^4, where T_s is the surface temperature. What equilibrium surface temperature do you expect with and without the snow layer? How long does it take to reach equilibrium? Plot a time series of the heat transfer from the ocean into the ice, and use this to estimate the rate at which the ice thickness would grow if you hadn't artificially kept the thickness fixed. *Python tips:* The basic solution of the initial value problem for the diffusion equation is implemented in the Chapter Script `diffusion.py`.

Problem 7.1.20 Suppose that the temperature at the subsolar point on the Moon is initially in equilibrium at $370K$. Suddenly, a lunar eclipse occurs, which eliminates the input of solar radiation. Use the diffusion equation to describe the temperature drop over the course of the next several hours, assuming (unrealistically) that the initial state is isothermal at $370K$. As a crude estimate of the night-time temperature of the Moon, compute how much the point would cool down if it remained unilluminated for $14d$. As a somewhat more realistic model of lunar cooling, assume instead that the subsurface is isothermal at $370K$ only down to a depth h, and that below that it is isothermal with a temperature of $250K$. Describe how the time series of temperature over $14d$ depends on h. You can estimate an appropriate value of h for the actual Moon by estimating how far heat diffuses into the lunar regolith in the course of a Lunar day (about $14d$). What is this depth?

Problem 7.1.21 *Diurnal cycle on rocky airless planets*

Solve the diffusion equation for a thick layer of the crust of a nearly airless rocky planet, subject to radiation from the surface at a rate σT_s^4 and insolation by the actual diurnally and seasonally varying time series of insolation at a latitude of 45^o. You may assume an insulating boundary condition at the lower boundary of the rock layer. If you make the layer thick enough, your result will not depend on the layer thickness. Based on the basic behavior of heat diffusion, how thick do you think the layer needs to be? Test your estimate against the actual behavior of your numerical calculation. For the purposes of this problem, you should put in numbers corresponding to the solar constant, obliquity, albedo, length of day and length of year corresponding to Present Mars, but you may assume the orbit to be circular. You may assume a constant diffusivity corresponding to the dry rock of your choice.

How does the amplitude of the surface diurnal cycle compare to the surface seasonal cycle? At what depth do you see a significant diurnal cycle? At what depth do you see a significant seasonal cycle? At what depth is the temperature essentially constant over the course of the year? How does the deep subsurface temperature compare with the surface temperature that would be in equilibrium with the annual mean absorbed solar radiation? Why do the two numbers differ?

Problem 7.1.22 Jupiter's moon Europa has essentially zero obliquity. It shares Jupiter's year and solar constant, and since it is tide locked to Jupiter its day is equal to its orbital period about Jupiter, or $3.55d$. Jupiter's orbit is essentially circular. Europa's crust is made largely of water ice, and there is essentially no atmosphere. Its albedo is about 0.67. Numerically solve the diffusion equation for a point on the equator of Europa, and describe the time series of surface and subsurface temperature. What is the deep-ice temperature? You may use an insulating lower boundary condition.

7.1.4 Orbital mechanics and eccentricity

Problem 7.1.23 The planet Gliese 581d has an orbital period of 66.8 d ($1d \equiv 86400s$). It's eccentricity is 0.38, and the parent star has a mass of .31 times that of the Sun. What is its semimajor axis? How does this compare with the result you would get assuming the planet to be in a circular orbit? The luminosity of Gliese 581 is 0.013 times that of the Sun. What is the instantaneous incident stellar flux (the "solar constant") seen by the planet at periastron? At apastron? What is the annual mean?

Do the same for the planet HD222582b, which is a 5 Jupiter-mass gas giant in orbit about a class G star having a mass and luminosity approximately equal to that of the Sun. The eccentricity of this planet is 0.76 and the orbital period is $572d$.

Problem 7.1.24 Discuss the effect of the precessional cycle on the annual mean solar flux distribution for a planet in an eccentric orbit. Specifically, show the solstice and equinox flux factors for a planet having 25^o obliquity, at several different stages of the precessional cycle. You may hold the eccentricity fixed at 0.1. For Earth's present obliquity and eccentricity (23.45^o and 0.17) how much does the precessional cycle affect the annual mean incident solar flux at $65N$ latitude?

Problem 7.1.25 it Summer melt energy

The *Summer melt energy* is defined as the incident solar flux at a given point, integrated over the period of time for which the flux exceeds some threshold value F_o. This is meant to serve as an index of the amount of energy available to melt ice, and so F_o is chosen to approximate the flux needed to bring the ice surface to the melting point. The appropriate choice of F_o depends on the ice albedo, meridional heat transport, surface flux conditions and various other things. Given a suitable choice of F_o (perhaps guided by observations or by examination of simple or comprehensive climate models), the melt energy statistic is simple to compute and provides a valuable guide as to the trade-off between the Kepler's law effect on duration of seasons and the precessional effect on intensity of seasons. There are actually two variants of the melt energy floating around. In the original one, the entire flux is integrated over the period where F_o is exceeded. In a more recent alternate form, one only integrates the portion of the flux that is in excess of F_o, rather in the spirit of "heating degree days" used to estimate demand for home heating fuel. In the following, you should compute both variants and compare. Note that the melt energy concept can be useful for studying ablation of mountain glaciers as well as for large continental ice sheets.

Compute the summer melt energy at latitudes 65^o and 40^o as a function of the phase of the precessional cycle for planets in orbits having eccentricity 0.02,0.05, 0.1 and 0.5 . Try a variety of different values for F_o ranging from zero to the seasonal maximum. You may hold the obliquity fixed at 25^o for this problem, but if you are energetic you may wish to explore the behavior for both higher and lower obliquities; the high obliquity case is relevant to Martian Milankovic cycles.

Hint: You don't need to specify the stellar constant $L_{\circledast}$, since the melt energy statistic can be formulated in terms of thresholds on the nondimensional flux factor $f(t)$. To obtain values for any given planet, the resulting statistic is simply multiplied by the value of $L_{\circledast}$ appropriate to the orbit's semi-major axis.

Problem 7.1.26 Mercury has an orbital period of $87.967d$, and a stellar day of $58.646d$. These are almost exactly in the ratio 3:2, and the state represents one of the stable spin states for a planet in a fairly eccentric orbit. (Up until 1965, it was thought that Mercury was approximately tide-locked to the Sun). The current eccentricity of Mercury's orbit is 0.206. Mercury's obliquity is essentially zero.

Make a graph of the time variation of the longitude of the subsolar point over the course of several Mercury years, taking into account the modulations of orbital speed caused by eccentricity. Are there any times when the sun appears to ”go backwards?”

At any given latitude and longitude, the mean insolation over a stated period at that point is determined by the time average of $L_{\circledast}(t)\cos\zeta$, where $L_odot(t)$ is the time varying incident solar flux given by the orbital position; in taking the average, you must remember to set the flux to zero when the sun is below the horizon. Make contour maps of annual-mean insolation patterns for several successive Mercury years. Does the pattern repeat? Show the patterns for an averaging period of 3 Mercury years.

The surface temperature for illuminated regions is largely determined by the instantaneous position of the subsolar point.However, the subsurface temperature depends on longer term averages, and the temperature of dark regions depends on how recently and how strongly they were illuminated, since that determines how much time they have had to cool down.

Problem 7.1.27 *Quasi-tidelocked spin states*

A planet in an eccentric orbit cannot be truly tidelocked, since the angular rotation rate of the planet's spin about its axis is nearly constant while the angular velocity of the planet's orbit about the star increases as the planet gets closer to the star. However, such a planet can still enter a quasisynchronous spin state, in which the planet's rotation period is equal to its year. In such a state, the planet always presents the same face to its star at periastron. It has been conjectured that close-orbit eccentric extrasolar planets like Gliese 581c are in such a state. In a truly tidelocked configuration, the subsolar point is anchored to a single point on the planet's surface, whereas in a quasisynchronous spin state, the subsolar point wobbles east and west over the course of the orbit.

Discuss the pattern of insolation on a planet in a quasi-synchronous orbit, for various values of the eccentricity. By how many degrees of longitude does the subsolar point wobble? What portion of the equator never receives any illumination? What portion is perpetually illuminated? (Without loss of generality, you may put the planet's equator in the plane of the planet's orbit; obliquity is zero in this class of problems). Plot the behavior of the flux factor along the equator at the periastron, apastron and at two intermediate points in the orbit. Plot the annual mean flux factor. If you know how to make countour plots, make a series of maps showing the seasonal cycle of the flux factor over the course of the year.

Problem 7.1.28 Mercury is in a quite eccentric orbit, but its spin state is not quasisynchronous. Instead, it is locked in a *3:2 spin-orbit resonance*, which is just a fancy way of saying that it rotates about its spin axis three times for each two orbits about the Sun. The obliquity is essentially zero. Determine the time series of Mercury's insolation for a geographically fixed point on the equator. Mercury's eccentricity is 0.2. It is possible that eccentric extrasolar planets may also be in 3:2 spin states rather than being quasisynchronous.

Problem 7.1.29 The directory `MilankovicMars` in the Workbook Datasets directory for this chapter contains orbital parameters for Mars over the past 21 million Earth years, in increments of a thousand years. The data itself is in the file `INSOLN.LA2004.MARSe.txt`. Note that the precession angle in this file is given relative to the spring equinox, and needs to be shifted so it is measured relative to the northern hemisphere solstice as used in the text. Both the obliquity and precession angle in this dataset are given in radians.

Using the orbital parameter data, compute a time series of the diurnally-averaged Martian insolation at $65N$, $30N$ and and the equator over the past 21 million years; plot the annual maximum, annual minimum and annual mean values. Then compute a time series of temperature at these latitudes, assuming the albedo to be 20%, that the surface is in instantaneous equilibrium with the absorbed solar radiation and that the atmosphere transports no heat away from the surface. You can also ignore the greenhouse effect of the atmosphere. (These assumptions are reasonable so long as the atmosphere remains roughly as thin as it is at present). Plot and discuss the annual maximum,annual minimum and annual mean tempeatures, with particular attention to regions that may become warmer than the freezing point of water and to the relative warmth of the low latitudes vs. high latitudes. Although the seasonal cycle has large amplitude, the annual mean surface temperature is still a relevant statistic, since it determines the deep subsurface temperature. When the annual mean tropical region becomes colder than the poles, ice can be expected to migrate from the poles to the tropics.

Recompute the annual mean temperature by finding the temperature which is in equilibrium with the annual mean flux. How much does this differ from your previous result? Why does it differ? If Mars had a thicker atmosphere or a deep mixed ocean would you expect the two different ways of computing the annual mean to differ so much?

Note that with a thin atmosphere the Martian diurnal cycle will be extreme, just as is observed today. The diurnally averaged temperature still is relevant, because it is the temperature of a layer somewhat below the surface. However, computing the diurnally averaged temperature by balancing radiation against the diurnally averaged insolation results in some error in the mean temperature, because σT^4 is nonlinear. Estimate the magnitude of the error in terms of the diurnal temperature range ΔT, which you can take as a given. A more precise way to solve this problem would be to compute that actual diurnal cycle of temperature using the heat diffusion equation, as was done in Problem 7.1.21.

Problem 7.1.30 Earth orbital parameter variations

The subdirectory `Milankovic` in the Workbook Datasets directory for this chapter contains data on the variations of Earth's eccentricity, obliquity and precession angle over the past five million years, together with associated documentation. The orbital parameters are in the text file `orbit91`. Angles in this file are in degrees; the precession angle is defined relative to the spring equinox, and needs to be shifted to yield the angle relative to the northern hemisphere summer solstice as used in the text.

Using the data in this file plot a time series of the annual mean insolation at the equator, $45N$, $45S$, $65N$ and $65S$. Since oceans respond primarily to the annual mean,these give an indication of the response of the oceans to Milankovic forcing. Discuss the relative role of eccentricity, obliquity and precession in the variations of the fluxes.

Over land, Milankovic variations have a strong effect on the seasonal cycle of temperature. Compare the seasonal cycle of diurnal mean insolation at $65N$ and $15N$ between the present, 6000 years ago (the Holocene Thermal Maximum) and 11000 years ago. The differences are quite subtle, so you can make the variations more apparent by plotting your results as deviations from the modern cycle.

Python tip The routine `getOrbit(...)` in `FluxExplorer.py` retrieves the orbital parameters for you, and takes care of shifting the precession angle so that it is measured relative to the solstice.

7.1.5 Simulation of planetary seasonal cycles

Python tip: The required calculations in this section can be done by modifying the Chapter Script `SeasonalCycle.py`.

Problem 7.1.31 Reproduce the calculation shown in Fig. **??** and compare the results with the analytic solution based on sinusoidal time-dependence of the insolation. Then re-do the numerical calculation with the assumed dynamical heat fluxes zeroed out. Put the dynamical heat fluxes back in, and re-do the calculation with a mixed layer depth of $5m$. Finally, for a $50m$ mixed layer depth see what happens if you double and halve the length of year.

You may assume the orbit to be circular throughout this problem.

Problem 7.1.32 *Southern hemisphere atmospheric seasonal cycle*

The Earth's southern hemisphere is mostly water, and because of the great thermal inertia of the ocean, to a first approximation the sea surface temperature can be regarded as nearly fixed in the course of the seasonal cycle. However, the atmosphere has lower thermal inertia, so that absorption of solar radiation by the atmosphere can drive a significant atmospheric seasonal cycle even if the sea surface temperature is fixed. This situation serves as a prototype for seasonal cycles on all waterworlds.

Explore the seasonal cycle in this regime by formulating and numerically solving the differential equation for the seasonal cycle of air temperature making the following assumptions: (a) The surface temperature remains fixed at a specified value T_s, (b) The atmosphere absorbs 20% of the incident solar energy, and the rest is either reflected back to space or passed on to the ocean surface, (c) The latitude is $45S$ and at this latitude the horizontal dynamical heat export from the atmospheric column vanishes, (d) The atmospheric temperature structure remains on the moist adiabat and (e) The OLR is given by the polynomial fit in Table **??** for Earth conditions with a CO_2 concentration of $300ppmv$. Try various values of T_s ranging from near freezing to $310K$, to explore the effect of increasing evaporation. You may assume the orbit to be circular if you like. Ideally, the thermal inertia of the atmosphere should be computed taking the latent heat storage into account (see Problem 7.1.15), but to keep things simple you may compute the atmospheric thermal inertia as if it were on the dry adiabat i.e. $\mu = c_p p_s / g$.

A peculiarity of the formulation of this problem is that T_s has been specified independently of CO_2. In general, the top-of-atmosphere fluxes will be out of balance even in the annual mean. Compute the annual mean top-of-atmosphere imbalance for each of your cases. What effect does the imbalance have on the mean atmospheric temperature? Try to rectify this problem by adjusting the CO_2 for each T_s so as to bring the annual mean top-of-atmosphere budget into approximate balance

Hint The atmosphere can be treated like a mixed layer ocean with an equivalent thermal inertia. The mixed layer equations only need to be modified to allow for the fact that only a portion of the solar flux is absorbed in the atmosphere, and to allow for the fact that the atmosphere exchanges energy with the surface as well as cooling to space. The latter fluxes can be computed using the linearized or fully nonlinear surface flux formulae given in Chapter **??**. (See also Problem 7.1.16) In computing the surface fluxes, assume a fixed surface wind speed of $5m/s$

and a roughness length appropriate to the ocean. You may use neutral boundary layer theory and ignore Monin-Obukhov stable boundary layer effects.

Problem 7.1.33 *Temperature response to Earth Milankovic cycles*

This problem explores the response of the Earth's temperature over atmosphere and ocean to Milankovic variations in insolation. The problem is treated through driving a mixed layer model with the actual seasonal cycle of insolation taking into account the variations in obliquity, eccentricity and precession angle. All of the calculations should be done using the polynomial OLR fit for pre-industrial CO_2 values (280 $ppmv$) so as to focus on Milankovic effects; in reality, the glacial-interglacial CO_2 variations also contribute to temperature variations. For oceanic cases, use a mixed layer depth of 50 m. As an approximation to thermal inertia over land on the seasonal time scale, take a mixed layer depth of 1 m. Assume the oceanic albedo to be 0.2 and the land albedo to be 0.3. You may assume that the atmospheric heat budget is in equilibrium, so that the net radiative heat loss from the column is given simply by the OLR. In the oceanic cases you may ignore the effects of sea ice formation, but you should note cases in which sea ice can form.

To attack this problem first write a function that takes obliquity, eccentricity and precession angle as inputs, and returns the annual mean, annual maximum and annual minimum temperature by solving the differential equation describing the mixed layer model.

Using this function, plot the annual maximum, minimum and mean temperatures for the past two million years at increments of one thousand years, using the time series of orbital parameters for the past. Make plots showing the behavior at latitudes of 60^o, 45^0 and 30^o, for land and oceanic conditions. In order to keep the mean temperature in a realistic range, you will need to impose a dynamical heat export from the atmospheric column. Assume that this has a fixed value of $-20W/m^2$ at $\pm 30^o$ latitude, zero at $\pm 45^o$ latitude and $40W/m^2$ at $\pm 60^o$ latitude. Discuss your results. To what extent do Milankovic cycles affect the annual mean temperature? To what extent do they affect the amplitude of the seasonal cycle?

Note that for each set of orbital parameters you only need to run the simulation for a few years until the seasonal cycle settles down. You do not need to run a simulation with time-varying orbital parameters over the full two million year period!

See Problem 7.1.30 for information on the dataset of Earth's Milankovic orbital parameter variations.

Problem 7.1.34 *Extreme Orbits*

Is high obliquity or high eccentricity a threat to habitability? This problem shows that a mixed layer ocean is in fact quite effective at averaging out extreme seasonal variations of insolation, thus keeping the temperature variations from getting too big. For each of the configurations stated below, numerically solve the mixed layer ocean seasonal cycle equations and show the time series of temperature at the equator, $\pm 45^o$, and the poles. Assume a mixed layer of depth $50m$ and a uniform albedo of 0.2 . For radiative cooling use the OLR polynomial fit in Table **??** for $1000ppmv$ of CO_2. You may neglect the effects of sea ice formation, but you should note and discuss conditions under which ice would form. Carry out your calculations with $L_{\circledast} = 1400W/m^2$; for elliptical orbits, this should be interpreted as the flux at a distance from the star equal to the semi-major axis of the orbit. Use the diurnally averaged form of the flux factor valid for planets that are rapidly rotating relative to the length of the year.

To carry out the simulation you will also need to make an assumption about the dynamical heat flux ΔF exported from the column. As a very simple representation of this, assume $\Delta F = a_F \cdot (1-\alpha) L_{\circledast} \cdot (\frac{1}{4} - \bar{f}(\phi)$, where $\bar{f}(\phi)$ is the annually averaged flux factor at the latitude in question and a_F is a constant between zero and unity measuring the effectiveness of the horizontal mixing of heat. When $a_F = 0$ each latitude is locally in equilibrium, and when $a_F = 1$ the annual mean temperature is nearly uniform over the globe. Carry out your calculations with $a_F = 0.5$, but feel free to explore the effects of varying this parameter.

Carry out your calculations for two different cases: (a) a planet with a circular orbit having an obliquity of 70^o, and (b) a planet with 25^o obliquity in an orbit with eccentricity 0.4.

If you are energetic, you can also explore the effect of variations in $L_\circledast$. How large does it have to be in order to suppress ice formation? At what point do you approach a runaway greenhouse threshold? How low does it have to be in order to form ice at 45^o latitude (a situation that is likely to provoke a Snowball once ice-albedo feedback is factored in)? Note that when there is no ice the only nonlinearity in the problem comes from $OLR(T)$, so in circumstances where this function can be linearized without loss of accuracy, all temperatures simply scale in proportion to $L_\circledast$. Discuss the effects of the deviation from linearity.

7.1.6 A little data Analysis

Problem 7.1.35 *Earth seasonal timeseries*

The subdirectory `NCEP` of the Workbook Datasets directory for this chapter contains the file `SeasonalTseries.txt`, which provides a 40 year record of monthly mean temperature at selected points on the Earth's surface.

Plot up and take a look at the two oceanic time series in the dataset. Model them using a mixed-layer model driven by the actual insolation computed for these location and using the polynomial OLR fit for $300ppmv$ CO_2 concentration. Optimize your fit between model and data by varying the mixed layer depth, albedo, and the amount of heat exported from the column by dynamic horizontal heat transports.

Since the dataset covers several decades, it gives you the chance to see how much the seasonal cycle varies from one year to the next Carry out your analysis for a few individual years of your choice, seeing how much your best-fit model parameters need to be varied. Also, you can form composite seasonal cycles for various 10-year periods by averaging all the Januaries, all the Februaries, etc. within the period.

Problem 7.1.36 The data file described in Problem 7.1.35 also contains a time series from the Australian interior desert and one from a high latitude Eurasian continental interior. Take a look at these, and see if you can make sense of the behavior in light of what you know about seasonal cycles over land. In the Eurasian case, can you see any imprint of thermal inertia due to the energy needed to melt snow in the springtime?

Problem 7.1.37 An interesting way of showing the effect of thermal inertia is to do a scatter plot of the surface temperature against the time series of insolation at the same point. If there were no thermal inertia, the temperature would just be a function of the insolation and the data would lie along a curve enclosing essentially no area. The "opening" of the loop shows the amount of memory in the system (both due to local and remote effects.

Carry out this analysis for the monthly mean temperature time series described in Problem 7.1.35. To connect the observations with theoretical expectations, plot some output from a mixed layer model with various layer depths in the same way; use the same insolation time series as you used in the data analysis.

Problem 7.1.38 Viking Mars temperature analysis

Data from the Viking landers VL1 and VL2 can be found in the `MarsViking` subdirectory of the Workbook Datasets directory for this chapter. VL1 was located at $22.48N$ and VL2 sat at $47.97N$.

Pick a short winter period and a short summer period (of a few days each) for each lander, and see how well you can model the diurnal cycle and mean temperature using an equivalent mixed-layer model to represent the thermal inertia of the solid surface. Drive the model using a time series of actual Martian insolation at the lander sites, computed from the present orbital parameters of Mars and formulae in the text. Optimize your fit by varying c_pH and the surface albedo. You may represent the surface cooling as being purely due to infrared emission at a rate σT_g^4. There will be some error in the results due to the neglect of the Martian greenhouse effect. There will be some additional mismatch since Viking measured the air temperature at lander height, which can be somewhat different from the surface temperature itself.

Then, if you are feeling ambitious, re-do the calculation using the diffusion equation in the solid surface in place of an equivalent mixed layer model. In this case, you vary the thermal conductivity to optimize the fit rather than c_pH.

Chapter 8

Atmospheric evolution modelling problems

8.1 Workbook

8.1.1 Silicate weathering

Problem 8.1.1 Based on the information shown in Fig. ??, how high does the temperature have to be in order to inhibit the formation of iron carbonate, if the surface silicate rocks are in contact with an atmosphere having a CO_2 partial pressure of $100mb$? *Python Tips:* The Chapter Script `EbelmenUreyEq.py` carries out the calculations needed for Fig. ??.

Problem 8.1.2 It has been estimated that the current CO_2 outgassing rate on Earth is about $8 \cdot 10^9 Mole/yr$. Assuming that this is entirely balanced by reaction with $CaSiO_3$ to form carbonate, and that suitably weatherable rocks have a surface area amounting to $\frac{1}{6}$ of the Earth's surface, how thick is the layer of rock that needs to be weathered away over the course of a thousand years?

The density of $CaSiO_3$ is about $3000kg/m^3$. In reality, it is a fairly uncommon mineral in the Earth's crust, and weathering reactions involving the Feldspar famiily of silicates are far more important. One member of this family is *Anorthite* ($CaAl_2Si_2O_8$), with density of $2700kg/m^3$. How would your result change if this were the mineral weathering? (As a crude approximation to the chemistry, assume that each C from a CO_2 molecule substitutes for one Si in the weathering products. The actual weathering products include a bewildering array of silicate-bearing clay minerals in addition to bicarbonate ion, which reduces the efficiency of the weathering reaction.)

Problem 8.1.3 *Artificial CO_2 outgassing*

Our own planet at present is plagued with too much CO_2 entering the atmosphere, owing to fossil fuel combustion. However a planet which had cooled off to the point that tectonics had stopped and outgassing of CO_2 had ceased would have the opposite problem, in that CO_2 would be taken out of the atmosphere by silicate weathering without being replaced by a source. This would lead to global cooling, and ultimately a snowball state on a planet with an ocean. How hard would it be to sustain a technological replacement for outgassing?

The present weathering rate on Earth consumes about $5 \cdot 10^{-5} Mole/m^2yr$ of carbon over continental crust. The weathering rate should be similar for any vegetated planet with an Earthlike climate and surface mineralogy. Suppose there is a fossil fuel reservoir containing $4000Gt$ of carbon. How long could the silicate weathering be offset by judicious burning of this fuel? How many kilowatt-hours of electricity would be produced by doing so? Compare that with current world electricity production.

But what can be done after the fossil fuels are gone? All is not lost. Use the equilibrium properties of the Ebelmen-Urey reactions to develop a strategy for cooking CO_2 out of various carbonate rocks by using solar energy to raise the rocks to some high temperature, and then letting them cool down. This could be done using mirrors to focus solar energy. You may assume that at sufficiently high temperatures the reaction proceeds so quickly that the time to reach equilibrium is not a significant limitation. Determine how many kg of $CaCO_3$ rock would have to be processed over each square meter of land each year. Estimate the energy required to do the processing. Compare this with a rough estimate of the solar energy falling on the area in the course of the year. What proportion of the solar energy needs to be tapped in order to save this planet from freezing? How might your answer differ if the rocks being processed are iron or magnesium carbonates instead?

The specific heat of calcium or magnesium carbonate is about $1000J/kgK$, and iron carbonate (siderite) is only slightly less at $700J/kgK$.

Problem 8.1.4 Re-do the calculation of Fig. **??** for a range of different outgassing rates relative to the present value W_o. How much would the outgassing rate have to be increased in order to make the Early Earth temperature similar to the present value?

Python tips: This and subsequent silicate weathering thermostat problems can be done by modifying the Chapter Script `WHAK.py`.

Problem 8.1.5 *Precipitation limitation and silicate weathering*

In Section **??** we showed that at sufficiently high temperatures the precipitation rate depends explicitly on the surface absorbed solar radiation, and tends to level off at a limiting value. Idealize this precipitation-temperature relation by assuming that the precipitation increases linearly with temperature at the rate assumed in the text until the implied surface latent heat flux equals the surface absorbed solar radiation, and thereafter remains constant.

For simplicity, we'll assume that the atmosphere does not absorb any shortwave radiation, though it may reflect some back to space before it can reach the ground, and we'll also assume that the albedo of the surface is zero, so that it absorbs all the shortwave radiation incident upon it. In this case, the atmospheric albedo is also the planetary (i.e. top-of-atmosphere) albedo, so the absorbed solar radiation at the surface or top-of-atmosphere is simply $\frac{1}{4}(1-\alpha)L_{\circledast}$. Compute the clear-sky OLR using the polynomial OLR fit for an Earthlike atmosphere.

First, assuming that the atmosphere does not absorb or reflect any incoming shortwave radiation, plot the temperature as a function of $L_{\circledast}$ for a range of different outgassing rates relative to Earth's present value. Discuss how these results differ from the case in which precipitation can increase indefinitely, paying particular attention to the implications for planets in orbits with low values of $L_{\circledast}$.

Now suppose that the atmosphere has clouds in it. If the clouds are low, they just change the albedo, and the effects are equivalent to just changing the value of $L_{\circledast}$. As an opposite extreme, suppose that the clouds have a height such that their albedo effect is exactly canceled by the reduction in OLR caused by the clouds, much as happens for deep clouds in the Earth's tropics today. The clouds will then leave the temperature unchanged for any given CO_2 concentration, but they will alter the precipitation. What does this effect do to climate once silicate weathering comes into equilibrium?

Carry out these calculations both for the biotic-land case in which the weathering rate is not directly dependent on pCO_2, and the abiotic case in which the weathering has a power-law dependence on pCO_2.

8.1.2 Ocean-Atmosphere partitioning

Python tips: The carbonate/bicarbonate equilibrium problems in the following group can be done with the assistance of the Chapter Script `CarbonateEq.py`.

Problem 8.1.6 Suppose space aliens suddenly and instantaneously steal all the O_2 from our atmosphere. Photosynthesis would eventually regenerate the oxygen, but how much O_2 would be put back into the atmosphere by outgassing from the ocean inventory alone? Give your answer in the form of the partial pressure of O_2. You may assume that O_2 is nonreactive in the ocean, so that the ocean storage is governed entirely by Henry's law.

Problem 8.1.7 Suppose a mutant nitrogen-fixing bacterium takes over the Earth and converts all the N_2 in the Earth's atmosphere into NH_3. Using Henry's law, determine the new composition of the atmosphere once it comes into equilibrium with a $50m$ oceanic mixed layer. What is the composition if the atmosphere comes into equilibrium with the top kilometer of ocean instead?

Problem 8.1.8 It has been proposed that Early Mars could have been warmed by a fairly high abundance of SO_2 in the atmosphere, arising from volcanic outgassing in an anoxic atmosphere. Suppose that Mars at that time had a $500m$ deep water ocean covering 20% of the surface area of the planet, with a mean temperature of $280K$. Use Henry's law to determine the partitioning of SO_2 between atmosphere and dissolved gas in the ocean. Ignoring any other oceanic SO_2 reservoir or sinks of SO_2, how many $Moles$ of SO_2 need to be outgassed in order to achieve a concentration of $100ppmv$ in a $1bar$ CO_2 atmosphere? Compare the estimated outgassing rate per unit surface area to the outgassing rate of CO_2 on Earth.

If you are ambitious, you may want to think about the circumstances under which sulfite and bisulfite can make up a substantial part of the oceanic sulfur pool.

Problem 8.1.9 *Linearized carbonate equilibrium*

Linearize the carbonate/bicarbonate equilibrium equations about an initial equilibrium state with atmospheric partial pressure $(pCO_2)_o$,bicarbonate concentration $[HCO_3^-]_o$, carbonate concentration $[CO_3^{--}]_o$ and a pH value $(pH)_o$. Specifically, write $pCO_2 = (pCO_2)_o + \Delta_o$, $[HCO_3^-] = [HCO_3^-]_o + \Delta_b$ and so forth, and solve for each perturbation Δ in terms of Δ_o assuming the perturbations to be small compared to the unperturbed values. Find an explicit expression for the change in pH and net atmosphere-ocean carbon storage as a function of the change in pCO_2. Consider both the case in which $[Ca^{++}]$ remains fixed at its unperturbed value and the case in which it is kept in equilibrium with a reservoir of solid $CaCO_3$.

Problem 8.1.10 Using Newton's method, write a routine to determine the pH which satisfies the charge balance relation in Eq. **??**, given pCO_2, $[Ca^{++}]$, and the equilibrium coefficients K_H, K_1 and K_2. Assuming $[Ca^{++}] = 0$, find the pH of sea water at $298K$ in equilibrium with atmospheric CO_2 concentrations of $280ppmv$ and $380ppmv$. Do the same for fresh water, for which $K_1 = 7.65 \cdot 10^{-9}$ and $K_2 = 1.01 \cdot 10^{-12}$ for concentrations measured as mole fraction. Find the pH of carbonated water in a sealed bottle, in equilibrium with 2 bar of pure CO_2 gas.

Problem 8.1.11 In this problem you will compute the equilibrium carbon storage in the ocean mixed layer as a function of the atmospheric pCO_2, assuming that there is no time for additional $CaCO_3$ to dissolve beyond what is already in the mixed layer at some initial time. You may assume the mixed layer to have a uniform temperature of $20C$.

First estimate the number of $Moles$ of H_2O in the mixed layer, assuming a $50m$ mixed layer depth covering $\frac{2}{3}$ of the Earth's surface. Next solve the carbonate/bicarbonate equilibrium equations and use charge balance to compute the concentration $[Ca^{++}]$ assuming that the mixed layer is in equilibrium with an atmosphere having CO_2 concentration 280 $ppmv$ (28 Pa partial pressure) and its pH is 8.15.

Next, compute the total atmosphere/mixed-layer carbon storage and the ocean pH as a function of pCO_2, holding $[Ca^{++}]$ fixed at the value computed in the preceding part. Use your results to discuss how much CO_2 from fossil fuel burning would remain in the atmosphere over times scales for which the mixed layer does not significantly exchange water with the deeper ocean.

Problem 8.1.12 *Where did my trillion tonnes go?*

In the text we computed where the first trillion tons of fossil-fuel carbon would go, over time scales long enough for surface waters to mix with the deep ocean, but too short for significant additional input of dissolved $CaCO_3$. What happens when additional carbon is added? How much does the reduction in oceanic pH reduce the ocean's ability to take up additional carbon?

To address this problem, compute the total atmosphere-ocean carbon content as a function of the atmospheric pCO_2, assuming $[Ca^{++}]$ to remain fixed at a pre-industrial value corresponding to equilibrium with an atmosphere having a CO_2 concentration of 280 *ppmv*. Present your results in the form of a graph showing the atmospheric CO_2 concentration as a function of the amount of fossil-fuel carbon released in Gt, showing results out to 5000 Gt (the estimated amount of carbon existing in worldwide coal deposits).

The outline of the calculation for this problem is essentially the same as for Problem 8.1.11, except that this time one equilibrates with the entire volume of the ocean rather than just the mixed layer.

Problem 8.1.13 Suppose that during a Snowball Earth episode,the atmospheric CO_2 concentration builds up to 10%. The ocean is in equilibrium with the atmosphere, and the dissolved calcium carbonate has been maintained at saturation. Since there is little input of carbonate from land weathering during a global glaciation, all of the calcium carbonate needed to maintain saturation needs to have come from seafloor dissolution. Estimate the number of $Moles$ of $CaCO_3$ per square meter of ocean floor that need to have dissolved in order to maintain saturation. How thick a layer of $CaCO_3$ does this correspond to? (The density of $CaCO_3$ is $2700kg/m^3$).

Problem 8.1.14 *Ecopoeisis on Mars*

Plot spoiler warning: Read the story *Ecopoeisis* by Geoffrey Landis before reading the rest of this problem.

In Landis' story, an ecosystem has been introduced on Mars which has increased the annual mean surface pressure of a pure CO_2 atmosphere to $100mb$. This has warmed the planet sufficiently to melt the glaciers and create a water ocean at one of the poles. Taking the temperature range indicated in the story as a given (i.e. leaving aside the question of whether $100mb$ of CO_2 would warm Mars that much) consider the effects of the polar ocean on the seasonal cycle of CO_2, including the effects of carbonate/bicarbonate equilibrium. At what parts of the seasonal cycle is the ocean taking up CO_2? At what point is it outgassing? What is the total mass of CO_2 outgassed each year? What is the effect on the surface pressure if the CO_2 remains in a column over the ocean? If the CO_2 is uniformly distributed over the planet?

If the ocean becomes supersaturated during the outgassing season and then releases all of the CO_2 in a ten day long burst at the end of the warm season, estimate the wind speed if the gas comes out in a radial current 3 km deep having uniform speed. Does the premise of the story work?

Hints: You will need to make an assumption about the mean pH, or equivalently the anion concentration $[Ca^{++}]$. The latter can be assumed constant over the course of the seasonal cycle (why?). Over much of the Martian surface the soils are acidic and carbonate-poor, so the mean pH could be quite low. On the other hand, some carbonate-rich soils have been discovered in polar regions, and so the ocean might be alkaline, with a pH similar to Earth's oceans. Discuss both of these limiting cases.

Assume that the ocean has a mass of $10^{19}kg$, spread uniformly over a layer centered on one pole, having area equal to 20% of the Mars surface. Consider the mixed-layer depth to extend through the full depth of the ocean. Model the seasonal ocean temperature cycle at the pole as a sinusoid with specified mean and amplitude. You are not expected to compute the seasonal cycle from a simulation, though in fact you are equipped to do so using the material covered in Chapter **??**. Try some different assumptions about the mean temperature and seasonal variation to see how this affects the basic premise of the story.

8.1.3 Atmospheric Chemistry

Problem 8.1.15 *Follow the photons I*

Consider a planet in an orbit about an M-dwarf star with photospheric temperature $3500K$, which has $L_{\circledast} = 2600W/m^2$ (about the same as Venus). Suppose that the planet is undergoing a runaway greenhouse and has developed a massive pure water vapor atmosphere. Assuming that water vapor is dissociated by Far UV photons with wavelengths shorter than $0.21\mu m$, estimate the photon flux in this band by assuming that it is half the blackbody value (as for Earth), and determine how long it would take to dissociate all the water originally contained in a liquid water ocean of average depth $2km$. Does the answer depend on the radius and surface gravity of the planet?

Assuming that the absorption cross section per molecule is $10^{-25}m^2$, estimate the thickness of the layer (expressed in Pa) over which the dissociation occurs,given a surface gravity of $20m/s^2$.

Problem 8.1.16 *Follow the photons II*

Consider a planet whose upper atmosphere consists of a mixture of gas A having initial molar concentration η_A and Far-UV cross section χ_A, with a gas B having initial molar concentration η_B and a smaller Far-UV cross section χ_B. The cross sections are independent of wavelength. The atmosphere is exposed to Far-UV with a given photon flux. The photolysis products do not react to re-form the photolyzed gases, and also do not absorb any Far-UV photons. Compute the time evolution of the concentrations of each gas under the following two limiting assumptions about mixing: (a) The top $10mb$ of the atmosphere is a mixed layer within which all molar concentrations are kept uniform in height as they change,and (b) There is no vertical mixing, so that a gas lost to photolysis is never replaced by mixing in from a layer where the gas is more abundant. Plot the time evolution of the profile of photon flux as well.

Put in some numbers corresponding roughly to an NH_3/CO_2 atmosphere on Early Earth, and discuss the extent to which a CO_2 dominated atmosphere could shield NH_3 from photolysis. The weakest points in your estimate will be the assumption that the cross-sections are wavelength-independent and the assumption that the photolysis products neither react nor absorb photons.

Problem 8.1.17 *CO_2 photolysis and reformation*

Re-do the CO_2 photolysis and reformation problem discussed in connection with Eq. **??**, but this time for a thick CO_2-dominated atmosphere in which mixing is slow enough that essentially none of the photolysis products reach the ground before reacting. Model the photolysis assuming a wavelength-independent cross section of $10^{-25}m^2$ for wavelengths shorter than $0.17\mu m$, and ignore absorption by the photolysis products. Model the mixing by assuming that a layer extending from the top of the atmosphere down to pressure p_{mix} is instantaneously mixed to uniform composition. Compute the molar concentrations of CO_2, CO and O once the system has reached equilibrium, as a function of p_{mix} and the photon flux. Restrict your attention to cases in which essentially none of the energetic photon flux exits the bottom of the mixed layer. Put in some numbers for the photon fluxes for Earth, Mars and Venus, and for a range of mixed layer thickness. For computing the rate constants, you may assume the temperature to be $200K$, and use the rate constant given in the text.

If you are ambitious, you can replace the mixed-layer assumption with a solution of steady-state diffusion equations for each of the chemical constituents, The methods used are precisely the same as those discussed in Chapter **??**, except for the need to introduce chemical sources and sinks. You can solve this either as a time-stepping partial differential equation problem or a steady state coupled second order ordinary differential equation problem. In either case, remember to take into account the different diffusion constants for the various species. With the diffusion formulation, you can also extend the problem to the case in which CO_2 is a minor constituent in an N_2 background gas (assumed transparent to FUV). In that case you would need to diffuse CO_2 as well as the photolysis products.

Note that even in the absence of catalysis by the OH radical, the reformation process would in reality be significantly affected by the recombination of $2O$ into O_2, and the photolysis of O_2.

8.1.4 Atmospheric Escape

Problem 8.1.18 Consider an atmosphere which is pure CO_2 below the homopause, but which photodissociates into CO and O at some pressure level p_1 which lies above the homopause. Note that the term "homopause" here refers to the level above which the mixing is dominated by diffusion rather than by turbulent motions, even though a single-component atmosphere will by definition have uniform composition everywhere; this is why some prefer the term "turbopause." For the purposes of this problem, you may ignore the recombination of the products back into CO_2. You may assume the atmosphere below the exobase to be isothermal at a temperature of $250K$. You may assume that conditions are such that the exobase lies at a low enough altitude that the gravity is essentially the same as the surface gravity, but you should state how the rest of your results depend on the planet's gravity.

First compute what the exobase altitude and density would be if the CO_2 didn't dissociate and the exobase were pure CO_2. In this case one expects a cool exobase temperature, which you can take to be $300K$. Then, keeping the exobase temperature the same determine the exobase height, density and composition as a function of the dissociation pressure level p_1. When the exosphere becomes dominated by O, the temperature is expected to rise. Say how your results change if you assume the exobase temperature to be $1000K$ instead.

Using the profiles of CO and O you computed above, discuss where you think the recombination into CO_2 is most likely to occur.

Problem 8.1.19 *Jeans escape from M-dwarf planets*

It has been suggested that for planets in close orbits about M-dwarfs, the high EUV output early in the life of the star could cause massive early atmospheric loss, which in the case of N_2 could be irreversible. One way this could happen would be for the EUV flux to be so high that the outer atmospheric temperature becomes very high, enhancing Jeans escape directly and through elevating the exobase to altitudes where the gravity is weaker. In this situation, it is only the outer EUV-heated part of the atmosphere would become high. Deeper parts of the atmosphere respond mainly to the net shortwave energy flux from the star, and would remain quite cool.

Consider a Super-Earth with radius and surface gravity twice that of Earth. The deep atmosphere, consiting of almost all of the mass of the atmosphere, is represented by an isothermal layer with a relatively cool temperature T_o; this layer is shallow enough that gravity is essentially constant within it. Above this lies an EUV-heated layer which begins at some specified pressure p_1 and has a high temperature T_{ex}. Supposing the atmosphere to be pure N_2 between p_o and the ground, dissociating to N at higher altitudes, compute the exobase density and altitude as a function of T_{ex}. How large does this temperature need to be in order for the reduction of gravity to be significant? Compute the Jeans escape flux, and determine how large T_{ex} needs to be in order to lose one bar of N_2 in a billion years. How does your answer depend on p_o?

Note that T_{ex} can substantially exceed the photospheric temperature of the star, since EUV radiation is nonthermal and comes in with an energy flux much greater than what would be expected from the photospheric blackbody temperature. Make an estimate of p_o in terms of the EUV photon flux using the number and absorption cross-section of molecules at altitudes above p_o, by estimating the number of molecules needed to use up the available photon supply. (Determining the temperature is much more complicated, since one needs to know something of how the N atoms in a nearly collisionless gas lose energy by radiation.)

Problem 8.1.20 *Diffusion-limited escape on Titan*

Estimate the diffusion-limit escape rate of hydrogen from Titan, assuming that hydrogen escapes by diffusion of H_2 through N_2 and then dissociates to H just before it escapes to space at the top of the atmosphere. Estimate how long it would take to lose one hydrogen atom from each CH_4 molecule in TItan's atmospheric CH_4 inventory, assuming that the photochemical reactions that split off the H are not rate-limiting.

Estimate the Jeans escape rate of H and compare it with the diffusion-limited rate, to see if the Jeans escape rate is high enough to make the escape diffusion-limited. Note that there are escape mechanisms that can be more effective than Jeans escape. To compute the Jeans escape rate you will need an estimate of the H mixing ratio at the exobase. This is difficult to do precisely, so a crude estimate of the likely range will suffice. You may assume that H is a minor constituent at the exobase.

Problem 8.1.21 In the text the diffusion limited escape problem was solved assuming that the removal from the exobase was so rapid that the concentration of the escaping constituent could be considered to be zero at the exobase. Re-do the problem using the actual flux boundary condition at the exobase, namely that $Ddn_A/dz = w_* n_a$ at the exobase, where n_A is the number density of the escaping consituent, D is the diffusivity of this constituent at the exobase and w_* is the escape flux coefficient, which has dimensions of a velocity. Show that it reduces to the result in the text in the limit of large w_*, and that the escape flux is limited by w_* rather than diffusion when w_J becomes small. Define the notions of "large" and "small" precisely, through formulation of a suitable nondimensional parameter.

In this problem, you may assume that the diffusive layer is thin compared to the radius of the planet, so that the acceleration of gravity can be considered constant.

Problem 8.1.22 *Speed of Sound*

First show that the wave equation $\partial_{tt}\phi - c^2\partial_{xx}\phi$ is satisfied by any function $\phi(x \pm ct)$, representing a disturbance propagating with speed c. Next, derive the wave equation for sound waves by showing that the momentum equation is $\partial_t u' = -\rho_o^{-1}\partial_x p'$ and the mass conservation equation is $\partial_t \rho' + \rho_o \partial_x u' = 0$ for small disturbances of velocity, density and pressure (u',ρ',p') of a state of rest with uniform density and pressure. Assume that pressure can be written as a function of density $p(\rho)$, and use the relation $p' = \rho' dp/d\rho$ to combine the two equations into a wave equation, from which you will be able to conclude that $c^2 = dp/d\rho$. Then, use the ideal gas equation of state and the definition of potential temperature to show that: (a) $c^2 = RT_o$ if the gas is constrained to remain isothermal, as in the case of very strong heat diffusion, and (b) $c^2 = \gamma RT_o$ if the flow remains adiabatic.

Problem 8.1.23 *The de Laval nozzle*

Consider compressible flow in a tube with circular cross section, whose radius varies with distance r along the axis of the tube. The velocity in the axial direction is uniform at each r, with value $u(r)$, so that the mass flux is $\rho(r)u(r)A(r)$. The fluid is not subject to any external heating, and the effect of gravity is negligible, as is the diffusion of heat. The gas satisfies the ideal gas equation of state. The equations describing this flow are identical to those used for spherically symmetric adiabatic hydrodynamic escape, except that $A(r)$ replaces $4\pi r^2$ and the gravitational potential has been dropped.

Solve the energy and entropy conservation equations to determine the characteristics of flow in this tube, given the conditions on u, ρ and T at the inlet. Show that if the inlet flow is subsonic, then the only way to cause the outlet flow to be supersonic is to arrange things such that there is a constriction of the tube such that the transonic point occurs at the position of the minimum of area. What do you think happens if an engineer tries to accelerate the flow to supersonic speeds using a conical nozzle with area decreasing monotonically toward the outlet?

Problem 8.1.24 *Adiabatic hydrodynamic escape from small bodies*

Consider a spherical body with a single-component atmosphere which is transparent to all incoming shortwave radiation as well as to outgoing infrared. It is a rocky body with a density of $3000 kg/m^3$. The surface is isothermal and has a specified temperature; once the temperature is given, it does not matter for the purpose of this problem whether the surface temperature is maintained by shortwave absorption, internal heat, or some combination of the two.

For what size of body can you sustain adiabatic hydrodynamic escape of H_2 if the surface temperature is $300K$? What answer do you get if the atmosphere is N_2 instead? Estimate the lifetime of the atmosphere as a function of the size of the body in each of these cases. For a G star with luminosity equal to the Sun, find the minimum size of body for which adiabatic hydrodynamic escape of an N_2 atmosphere can be sustained as a function of the orbital radius, assuming zero albedo of the body and that the surface is heated by shortwave absorption.

Problem 8.1.25 *Isothermal hydrodynamic escape flux*

Compute the isothermal hydrodynamic escape flux as a function of the density at the base of the escaping atmosphere and the critical (i.e. transonic) point position. State the corresponding temperature of the layer, which is the same as the critical point temperature since the flow is isothermal. The "base" of the escaping flow can be taken to be at any convenient point of reference where the kinetic energy of the radial velocity is negligible. Put in numbers corresponding to the escape of atomic hydrogen on Earth and on Venus, and say how the results depend on the assumed critical point position. What happens if you try to make a heavier species escape? Consider the cases of atomic oxygen on Venus and atomic carbon on Mars as an example of this. Think hard about the question of whether you can get around the requirement of very high base densities by moving the altitude you choose for the base outward. *Hint:* The answer is that you can'tf.

Use the entropy equation to derive an expression estimating how much radiative flux is needed to sustain the escape. How does the molecular weight of the escaping species affect the result? *Hint:* Recall that the specific heat is largely determined by molecular weight.

To complete the calculation of escape flux, you need to put in a number for the base density. One gets the same answer regardless of where one takes the base to be so long as the kinetic energy of the radial velocity is negligible there, since one is in the end solving for the structure using hydrostatic balance, whether that is done directly or via the radial momentum equation. Let's restrict attention to the H escape case. A precise calculation requires solving the full photochemistry problem simultaneously with a representation of vertical transport, but we'll take some shortcuts. To get the base density, assume that the homopause (i.e. turbopause) is pure H_2O, but that the water vapor decomposes into H_2 and O_2 just above the homopause. Then, assuming the temperature of the layer above the homopause to be the temperature you computed on the basis of the critical point position above, compute the scale height of each species, and find the altitude at which H_2 is 90% dominant, by Mole fraction. Assume that just above this point, the H_2 dissociates into H, and take the result as your base density of H.

Using this estimate, compute the escape flux for Venus, Earth and Mars, and determine how long it would take to lose the H in a $2km$ deep ocean for each planet. Estimate the EUV flux needed to sustain the escape, and compare this to the typical EUV fluxes encountered in the Solar System. Recall that the young Sun could have put out an EUV flux a factor of ten or more greater than is observed at present.

Problem 8.1.26 *Impact erosion basics*

First, reproduce the Mars and Earth lines in Table **??** to make sure you understand how the calculation of impact erosion is done. Then compute the analogous results for a 1 bar N_2 atmosphere on the Moon assuming a surface temperature of 260. Do the calculation first for the case where the Moon orbits the Earth in its present orbit, and then for the case in which the Moon orbits the Sun directly without sharing the orbit with Earth.

Then, re-do the estimates of m_{tot} for Earth and Mars assuming that the mass spectrum of impactors has the exponential form $n(m) \sim \exp(-m/m_1)$ instead of a power law. In this case, do you need to truncate the distribution at a maximum mass m_+?

Problem 8.1.27 *Alternate assumptions about impactor energy*

In the traditional view of impact erosion, the velocity of impactors scales with the escape velocity of the target planet. More recent models, particularly of the Late Heavy Bombardment, suggest that the population of impactors may instead be drawn from a pool of planetesimals in orbit beyond Jupiter, whose orbits are perturbed and become eccentric enough to cross the inner Solar system. This is known as the "Nice Model" (after the town in Southern France, though the model is indeed a nice one as well).

As a simple exploration of this situation, consider a target planet with mass M_p and radius a in a circular orbit at distance r_p from its star. The star has mass $M_\circledast$. The population of impactors are in eccentric orbits which take them from a maximum distance r_+ from the star to a minimum distance $r_- < r_p$, which can collide with the target planet when they cross its orbit. The impactors and the target orbit the star in ths same direction, so collisions will be rear-end rather than head-on. Use conservation of energy and angular momentum to find an expression for the impactor speed at the point where it crosses the target orbit at distance r_p from the star. Use the expression for the ellipse to find the angle at which the impactor's orbit crosses the planet's orbit, and then use this angle and the planet's orbital speed to determine the velocity of the impactor *relative* to the planet. Under what circumstances is this relative velocity large in comparison to the escape velocity of the planet (which gives the impact speed of an impactor initially at rest relative to the planet, dropped from a great height)? Put in some numbers corresponding to Earth and Mars.

Now re-do the derivation of the formula relating mass of atmosphere lost to net impactor mass, this time assuming the typical velocity of the impactor to have a fixed value v_i independent of the size of the target planet. This assumption is valid when the relative approach speed calculated in the preceding paragraph is large compared to the escape velocity of the planet. Assume the mass distribution of impactors to follow the same power law assumed in the text. Discuss how the tendency of a planet to lose its atmosphere scales with the size of the planet in this case. Note that if two planets are competing for the same population of impactors, then each will capture impactor mass in proportion to its cross-sectional area, so that the smaller planet will get less total mass of impactors than the larger planet.

Apply these results to the relative erodability of atmospheres on Early Earth and Mars, assuming the impactor population to have eccentric orbits extending from that of Jupiter to that of Venus.

Chapter 9

Meridional heat transport modeling problems

9.1 Workbook

9.1.1 Basic concepts

Problem 9.1.1 The pure CO_2 atmosphere of Mars currently has a surface pressure on the order of $10mb$. Recall that $\Phi/L_\circledast$ is the key nondimensional quantity determining the effect of meridional heat flux on the planetary temperature gradient. Making use of the scaling of Φ with planetary radius and atmospheric mass path p_s/g, how strong would typical Martian winds need to be in order for the meridional heat transport to have an effect on Martian temperature variations of similar magnitude to that in Earth's atmosphere? To carry out this estimate you need an estimate of ΔT. For that, you may assume that under the conditions where Mars is given an Earthlike temperature gradient, ΔT has a value similar to that on the present Earth. Don't forget to take into account the specific heat of CO_2 vs. that of Earth air.

You should do this estimate ignoring the effects of moisture in the Earth case. In reality, about half of the heat transport in the present Earth's atmosphere is due to moisture. What does this fact do to your estimate of the winds required on Mars?

9.1.2 Diffusive energy balance models

General hints: For many of the problems in the following group you will need the annual-mean flux factor as a function of latitude for a circular orbit. You can compute this by numerically evaluating an integral using the formulae given in Chapter **??** (or using the ready-made `Python` routines provided as part of the courseware). Alternately, to keep things simpler, you can make use of the fact that the flux factor is well fit by the polynomial $f(y) = a_o + a_2 y^2 + a_4 y^4$ where $y = \sin\phi$. The coefficients (a_o, a_2, a_4) have the values $(.311, -.0873, -.162)$ for 10^o obliquity, $(.308, -.134, -.0682)$ for 20^o obliquity, $(.300, -.155, .0116)$ for 30^o obliquity and $(.284, -.137, .0563)$ for 40^o obliquity.

To keep the energy balance model linear, you will also need a linearized form of the function $OLR(T)$. For an Earthlike atmosphere at 50% relative humidty with a CO_2 concentration of $300ppmv$, a reasonable fit to the actual OLR curve in the range of $220K$ to $310K$ is $OLR(T) = a + b\cdot(T - 220)$ where $a = 113W/m^2$ and $b = 2.177W/m^2K$. For other values of CO_2 concentration in the range between $50ppmv$ and $5000ppmv$, a fairly good approximation to reality can be obtained by changing the value of a to $a' = a - 5.77\ln(pCO_2/300)$, where pCO_2 is measured in $ppmv$.

Python tips: The basic steady-state linear diffusive energy balance model is implemented in the Chapter Script `SteadyLinearDiffusiveEBM.py`. Most of the numerical problems in the following group can be done by modifying this script.

Problem 9.1.2 For the general steady diffusive EBM described in Eq. **??**, show that in the limit of very small D, the temperature is determined by local energy balance. Show that in the limit of very large D the temperature is uniform. Compute this temperature, by considering the integral of Eq. **??** over the entire planet. Show that, when $OLR(T)$ is linear in T this procedure gives the exact correct result for the global mean T regardless of the magnitude of the diffusivity.

In nondimensional terms, what is meant by "small" diffusivity? ***Hint: build a number with the same dimensions as diffusivity, based on the typical pole to equator variation in solar absorption.***

Problem 9.1.3 Following on from the results of Problem 9.1.2, find an approximate expression for the variation of T over the globe when D is large but not infinite. Specifically, write $T(h)$ as $T_o + D^{-1}T_1(y)$ where T_o is constant, plug into Eq. **??**, collect terms of order D^{-1}, and analytically find an expression that solves the resulting equation. You should be able to do this problem without assuming that $OLR(T)$ is linear.

Problem 9.1.4 *Numerical solution for uniform albedo*

Numerically solve the steady linear diffusive energy balance model equations for a planet having uniform albedo of 0.2, forced by annual-mean insolation for a circular orbit. The planet has Earthlike atmosphere and gravity, so a linear $OLR(T)$ fit based on Earth conditions can be used. Explore the behavior as a function of obliquity, $L_\circledast$, and the atmospheric CO_2 concentration. Determine the conditions under which the planet can be expected to be ice-free, assuming it to be a waterworld. You make keep the large scale diffusion parameter D fixed at $1W/m^2$. Leaving aside the effects of gravity, how would the size of the planet affect the results (if at all)?

Hint: If you rewrite the model in terms of the shifted temperature variable $T' \equiv (T - 220 + a'/b)$ then $L_\circledast$ can be scaled out of the problem by using $L_\circledast/b$ as a temperature scale. Thus, results for arbitrary $L_\circledast$ can be obtained from a single numerical solution.

Problem 9.1.5 *Numerical solution for solstice conditions*

Problem 9.1.6 *Ice-albedo bifurcation revisited*

Using a steady, linear diffusive energy balance model with the ice latitude determined consistently with the freeze temperature as described in the text, recompute the ice-albedo bifurcation diagram using $L_\circledast$ as the control parameter, which we earlier computed using a zero-dimensional globally averaged model (Fig. **??**). For this problem, you may assume that the atmosphere is transparent to infrared radiation, but is nonetheless massive enough to transport significant amounts of heat. Explore the behavior as a function of the diffusivity used to represent large scale heat transport, and as a function of the obliquity of the planet, assuming the orbit to be circular and that the length of the day is short compared to the length of the year, as on Earth. Display your results in two different forms: first using the global mean temperature as the response variable (as in Fig. **??**) and then using the ice margin y_{ice} as the response variable.

You may assume that the freeze temperature of sea water is $271K$ and that the ocean albedo is 0.1. Show results for ice albedos of 0.5 and 0.7, corresponding to different degrees of snow cover on the ice.

To implement the calculation, you first write a routine that computes the ice-margin temperature as a function of the ice margin position, making use of the differential equation integration described in the text. Then, you hand this function to a Newton's method solver or some other root-finder, which will find the positions of the equilbria; you need to re-run the root-finder many times with different initial conditions in order to find all the solutions, since there may be more than one. One way to make sure you find them all is to generate guesses by computing a rather coarse array of $T_{ice} - T_{freeze}$ and look for the values of y where there is a zero-crossing.

Problem 9.1.7 *An EBM based on moist static energy diffusion*

Derive a steady energy balance model based on the assumption that the moist static energy flux is proportional to the gradient of mean moist staticc energy E_{atm} defined in Eq. **??**, instead of being proportional to the gradient of the zonal mean surface temperature T. In other words, derive an alternate energy balance model based on diffusion of moist static energy instead of temperature. In order to close the problem it is necessary to be able to write E_{atm} as a function of T, since the OLR (and perhaps also the albedo) is a function of T. Show that this can be done provided that the the following two conditions are satisfied: (1) The atmosphere is saturated everywhere, and

(2) The atmosphere is on the moist adiabat, so that the moist static energy is uniform in each column. State the function $E_{atm}(T)$ in terms of the saturation vapor pressure function.

Using the chain rule on the y derivative of $E_{atm}(T)$ appearing in the diffusion equation, show that the equation can be re-cast as a temperature diffusion energy balance model with a temperature dependant diffusivity. Discuss the behavior of this diffusivity as a function of temperature, paying particular attention to the cold, dry limit and the very hot moisture-dominated limit.

What kind of additional equation do you need to close the problem if the vertical temperature profile is described by the saturated moist adiabat, but there are nonetheless unsaturated regions? This situation is an accurate description of the Earth's tropics, and a somewhat worse approximation for the extratropics.

Throughout this problem, you may assume moisture is dilute, so that it is valid to use the approximate dilute form of moist static energy.